2.3 Systems of Linear Equations: Underdetermined and Overdetermined Systems

Example:

$$\begin{aligned} x+2y+z&=-2 \\ -2x-3y-z&=1 \\ 2x+4y+2z&=-4 \end{aligned} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & -2 \\ -2 & -3 & -1 & 1 \\ 2 & 4 & 2 & -4 \end{array}\right] \xrightarrow{\text{RREF}}$$

Now check is this in RREF?

look at this

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 4 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Are the rows of all zeros is below the non-zero rows? ✓

Is the first non-zero entry in any row is a 1? ✓

Do the leading 1's go down in a diagonal? ✓

If a column has a leading 1 then does the rest of the column have zeros? ✓

A column containing a leading 1 is called a *unit column* and the variable associated with the column is a *basic variable.*

$$\begin{array}{ccc} x & y & \cancel{z} \to t \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 4 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{aligned} x-t&=4 \\ y+t&=-3 \\ 0&=0 \end{aligned}$$

$x-t=4 \Rightarrow x=t+4$

$y+t=-3 \Rightarrow y=-t-3$

$(x,y,z)=(t+4,-t-3,t)$

t any $\mathbb{R}$

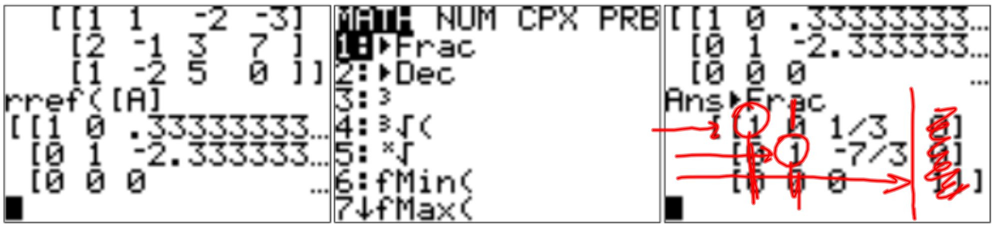
part. solns

$t=0$ $(4,-3,0)$

$t=1$ $(5,-4,1)$ etc

Example:

$$\begin{aligned} x+y-2z&=-3\\ 2x-y+3z&=7\\ x-2y+5z&=0 \end{aligned} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -2 & -3\\ 2 & -1 & 3 & 7\\ 1 & -2 & 5 & 0 \end{array}\right] \xrightarrow{\text{RREF}}$$



RREF form?

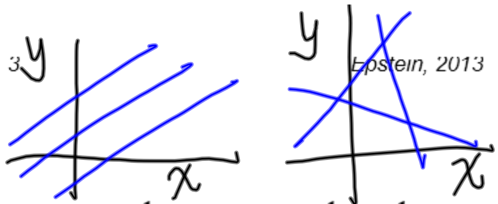
Are the rows of all zeros below the non-zero rows?
Is the first non-zero entry in any row is a 1?
Do the leading 1's go down in a diagonal?
If a column has a leading 1 then is the rest of the column is zero?

$$\begin{array}{cccc} x & y & z \to t & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 1/3 & 0\\ 0 & 1 & -7/3 & 0\\ 0 & 0 & 0 & 1 \end{array}\right]$$

$x + 1/3\, t = 0$

$y - 7/3\, t = 0$

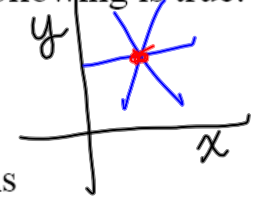
$0 = 1 \Rightarrow$ NO SOLN



NO SOLN

Number of Solutions Theorem

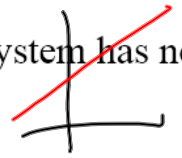
Case 1: If the number of equations is greater than or equal to the number of variables in a linear system then one of the following is true:



one soln

- The system has no solution
- The system has exactly one solution
- The system has infinitely many solutions

Case 2: If there are fewer equations than variables then the system has no solution or infinitely many solutions.



Example: Solve the following system (Case 2)

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 4x_3 & = & 2 \\ x_1 & + & x_2 & + & 2x_3 & = & 1 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 4 & 2 \\ 1 & 1 & 2 & 1 \end{array}\right] \xrightarrow{RREF}$$

$$\begin{array}{c} \begin{array}{cccc} x & y & \cancel{z} \to t & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 \end{array}\right] \end{array}$$

$x = 0$

$y + 2t = 1$

$y = 1 - 2t$

$(x, y, z) = (0, 1-2t, t)$

t any $\mathbb{R}$

Do RREF ⇒ error b/c more rows than coln

Example: Solve the following system (Case 1):

$$\left.\begin{array}{rcrcr} 4x & + & 6y & = & 8 \\ 3x & - & 2y & = & -7 \\ x & + & 3y & = & 5 \\ 2x & + & 6y & = & 10 \end{array}\right\}$$ try on your own

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A company is buying three kinds of vehicles. Carts hold 3 people and cost \$9,000, vans hold 8 people can cost \$27,000 and minivans hold 7 people and cost \$27,000. The company needs to seat 48 people and has \$162,000 to purchase vehicles. How many of each type of vehicle can be purchased?

x = # of carts
y = # of vans
z = # of m.vans

$3x + 8y + 7z = 48$ (seats)
$9000x + 27000y + 27000z = 162000$ (\$ spent)

RREF $\Rightarrow$
$$\begin{array}{cccc} x & y & \cancel{z}\ t & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 6 \end{array}\right] \Rightarrow \begin{array}{l} x - 3t = 0 \\ y + 2t = 6 \end{array} \Rightarrow \begin{array}{l} x = 3t \\ y = 6 - 2t \end{array}$$

$(x, y, z) = (3t, 6-2t, t)$ where t = # of m.vans

$t=0 \Rightarrow (0,6,0) \Rightarrow$ Buy 0 carts, 6 van, and 0 m. vans
$t=1 \Rightarrow (3,4,1) \Rightarrow$ " 3 " 4 " " 1 "
$t=2 \Rightarrow (6,2,2)$ 6 2 2
$t=3 \Rightarrow (9,0,3)$ 9 0 3

2.4 Matrices

A matrix is a compact way of organizing and displaying data.

A matrix is often denoted by a capital letter *M* or *A*.

A matrix having *m* rows and *n* columns is an *m* x *n* matrix

$M_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ is a 3x1 matrix

$M_2 = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix}$ is a 1x3 matrix

These two matrices are NOT EQUAL.

A matrix is called square if it has the same number of rows and columns.

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is a 2 x 2 square matrix.

$a_{11} = 1$ $a_{21} = 3$

$a_{12} = 2$ $a_{22} = 4$

a_{ij} is the element in the i^{th} row and j^{th} column of matrix *A*.

Example –
There are three stores. In the first week store I sold 88 loaves of bread, 48 quarts of milk, 16 jars of peanut butter and 112 pounds of cold cuts. At the same time, store II sold 105 loaves of bread, 72 quarts of milk, 21 jars of peanut butter and 147 pounds of cold cuts. Store III sold 60 loaves of bread, 40 quarts of milk, 0 jars of peanut butter and 50 pounds of cold cuts.

Organize this data in a 3 x 4 matrix.

$$\begin{array}{c} \\ \text{Store I} \\ \text{Store II} \\ \text{Store III} \end{array} \begin{array}{c} \begin{array}{cccc} B & M & PB & CC \end{array} \\ \begin{bmatrix} 88 & 48 & 16 & 112 \\ 105 & 72 & 21 & 147 \\ 60 & 40 & 0 & 50 \end{bmatrix} \end{array}$$

MATRIX ALGEBRA

Equality - two matrices are equal if and only if each pair of corresponding elements are equal.

Example - Find the values of a, b, c, d given

$$\begin{bmatrix} 1 & b \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} a & -1 \\ c & d \end{bmatrix}$$

$1 = a$ $b = -1$
$3 = c$ $0 = d$

Addition - two matrices are added by adding the pairs of elements in each location.

Example - find A+B where $A = \begin{bmatrix} 2 & -3 \\ 0 & 5 \\ 0.25 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} -0.25 & 7 \\ -3 & 1.5 \\ 9 & 2 \end{bmatrix}$

$3\times 2 + 3\times 2 = 3\times 2$

$$\begin{bmatrix} 1.75 & 4 \\ -3 & 6.5 \\ 9.25 & 8 \end{bmatrix}$$

Transpose - The transpose of a matrix is found by switching the rows and columns of the matrix.

If A is a 3x2 matrix then A^T will be a 2x3 matrix.

$$A = \begin{array}{cccc} & a_{11} & a_{12} & \\ a_{21} & \begin{bmatrix} 2 & -3 \\ 0 & 5 \\ 0.25 & 6 \end{bmatrix} & & \begin{array}{c} \\ a_{22} \\ a_{32} \end{array} \end{array}$$

3x2

$$A^T \begin{bmatrix} 2 & 0 & .25 \\ -3 & 5 & 6 \end{bmatrix}$$

2x3

[2] this is a 1x1 matrix

2 this is the number 2

Scalar multiplication –

A scalar is a number (NOT a matrix).

Multiply a matrix by a scalar by multiplying every element in the matrix by the scalar.

Example - find $-2A$.

$$-2A = -2\begin{bmatrix} 2 & -3 \\ 0 & 5 \\ 0.25 & 6 \end{bmatrix} = \begin{bmatrix} -4 & 6 \\ 0 & -10 \\ -.5 & -12 \end{bmatrix}$$

[A]

2.5 Multiplication of Matrices

Example

A flower shop sells 96 roses, 250 carnations and 130 daisies in a week. The roses sell for \$2 each, the carnations for \$1 each and the daisies for \$0.50 each. Find the revenue of the shop during the week.

$$R = 96 \times 2 + 250 \times 1 + 130 \times 0.5 = \$507$$

Express the number of flowers in a 1×3 matrix:

$$A = \begin{array}{cc} & \begin{array}{ccc} R & C & D \end{array} \\ \# & \begin{bmatrix} 96 & 250 & 130 \end{bmatrix} \end{array}$$

Next express the price as a 3×1 matrix:

$$B = \begin{array}{cc} & \$ \\ \begin{array}{c} R \\ C \\ D \end{array} & \begin{bmatrix} 2 \\ 1 \\ .5 \end{bmatrix} \end{array}$$

$$A \times B = \begin{array}{cc} & \begin{array}{ccc} R & C & D \end{array} \\ \# & \begin{bmatrix} 96 & 250 & 130 \end{bmatrix} \end{array} * \begin{array}{cc} & \$ \\ \begin{array}{c} R \\ C \\ D \end{array} & \begin{bmatrix} 2 \\ 1 \\ .5 \end{bmatrix} \end{array}$$

$$= \begin{bmatrix} 96 \times 2 + 250 \times 1 + 130 \times .5 \end{bmatrix} = \begin{bmatrix} 507 \end{bmatrix}$$

In general, if A is $1\times n$ and B is $n\times 1$, the product AB is a $\boxed{1\times 1}$ matrix:

$$A\cdot B=\begin{array}{cccc} x & y & & z \end{array}$$
$$A\cdot B=\begin{bmatrix}a_{11} & a_{12} & \dots & a_{1n}\end{bmatrix}\cdot\begin{bmatrix}b_{11}\\ b_{12}\\ \vdots \\ b_{n1}\end{bmatrix}=\begin{bmatrix}a_{11}\cdot b_{11}+a_{12}\cdot b_{21}+\dots+a_{1n}\cdot b_{n1}\end{bmatrix}$$

rows col row col

If A is an $m\times n$ matrix and B is a $n\times p$ matrix, then the product matrix $A\cdot B=C$ is an $m\times p$ matrix.

row col

$$A\cdot B=\begin{bmatrix}a_{11} & a_{12} & \dots & a_{1n}\\ a_{21} & a_{22} & \dots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & & a_{mn}\end{bmatrix}\cdot\begin{bmatrix}b_{11} & b_{12} & \dots & b_{1p}\\ b_{21} & b_{22} & \dots & b_{2p}\\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{np}\end{bmatrix}$$

use row 1 of matrix A
use col 2 of matrix B

$$=\begin{bmatrix}\overbrace{a_{11}\cdot b_{11}+a_{12}\cdot b_{21}+\dots+a_{1n}\cdot b_{n1}}^{(ab)_{11}} & (ab)_{12} & \dots & (ab)_{1p}\\ (ab)_{21} & (ab)_{22} & \dots & (ab)_{2p}\\ \vdots & \vdots & \ddots & \vdots \\ (ab)_{m1} & (ab)_{m2} & \dots & (ab)_{mp}\end{bmatrix}$$

Matrix multiplication is not commutative. In general, $AB \neq BA$

$(2\times 2) * (2\times 2) = (2\times 2)$

$(2\times 4) * (4\times 3) = 2\times 3$

$(4\times 3) * (2\times 4)$ NO

Example
Find the products AB and BA where

$$A = \underset{[A]}{\overset{2\times 2}{\begin{bmatrix} 1 & 0 \\ -2 & 3 \end{bmatrix}}} \text{ and } B = \underset{[B]}{\overset{2\times 2}{\begin{bmatrix} -1 & 2 \\ 0 & -3 \end{bmatrix}}}$$

$$BA = \begin{bmatrix} -1 & 2 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} -1 & 2 \\ 2 & -13 \end{bmatrix} \qquad BA = \begin{bmatrix} -5 & 6 \\ 6 & -9 \end{bmatrix}$$

One special matrix is called the identity matrix, I.
It is a square matrix with 1's on the diagonal and zeros elsewhere,

$$I = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

A is 4x4
AI means AI_4

I_2 is a 2×2 identity matrix and I_n is an $n\times n$ identity matrix.

```
NAMES MATH EDIT        identity(3)
1:det(                      [[1 0 0]
2:T                          [0 1 0]
3:dim(                       [0 0 1]]
4:Fill(
5:identity(
6:randM(
7↓augment(
```

The identity matrix has the following properties: $AI = A = IA$

Example

Cost Analysis - The Mundo Candy Company makes three types of chocolate candy: cheery cherry (cc), mucho mocha (mm) and almond delight (ad).

The candy is produced in San Diego (SD), Mexico City (MC) and Managua (Ma) using two main ingredients, sugar (S) and chocolate (C).

Each kilogram of cheery cherry requires 0.5 kg of sugar and 0.2 kg of chocolate.
Each kilogram of mucho mocha requires 0.4 kg of sugar and 0.3 kg of chocolate.
Each kilogram of almond delight requires 0.3 kg of sugar and 0.3 kg of chocolate.

(a) Put this information in a 2x3 matrix.

$$\begin{array}{c c} & \begin{array}{ccc} cc & mm & ad \end{array} \\ \begin{array}{c} Sug \\ Choc \end{array} & \begin{pmatrix} .5 & .4 & .3 \\ .2 & .3 & .3 \end{pmatrix} \end{array}$$

(b) The cost of 1 kg of sugar is \$3 in San Diego, \$2 in Mexico City and \$1 in Managua. The cost of 1 kg of chocolate is \$3 in San Diego, \$3 in Mexico City and \$4 in Managua.

A is (2X3)

Put this information into a matrix in such a way that when it is multiplied by the matrix in part (a) it will tell us the cost of producing each kind of candy in each city.

2 costs and 3 cities so ~~B_1 (2X3 matrix)~~

B_2 (3X2 matrix)

Answer has 3 cities and 3 candy so a 3X3

Dimensions

① $B_1 \cdot A$ is $(2\times3)\cdot(2\times3)$ NO

② $A \cdot B_1$ is $(2\times3)(2\times3)$ NO

③ $A \cdot B_2$ is $(2\times3)(3\times2) = 2\times2$ NO

④ $B_2 \cdot A$ is $(3\times2)(2\times3) = 3\times3$ yes?

S C match

$$\begin{array}{r} \\ \text{cost in SD} \\ \text{cost in MC} \\ \text{cost in MA} \end{array} \begin{array}{c} \text{S} \quad \text{C} \\ \begin{pmatrix} 3 & 3 \\ 2 & 3 \\ 1 & 4 \end{pmatrix} \end{array} \times \begin{array}{c} \\ \text{S} \\ \text{C} \end{array} \begin{array}{c} \text{cc} \quad \text{mm} \quad \text{ad} \\ \begin{pmatrix} .5 & .4 & .3 \\ .2 & .3 & .3 \end{pmatrix} \end{array}$$

$$= \begin{array}{r} \\ \text{SD} \\ \text{MC} \\ \text{MA} \end{array} \begin{array}{c} \text{cc} \quad \text{mm} \quad \text{ad} \\ \begin{pmatrix} 2.1 & 2.1 & 1.8 \\ 1.6 & 1.7 & 1.5 \\ 1.3 & 1.6 & 1.5 \end{pmatrix} \end{array}$$

$2.1 = \underbrace{\$3/\text{kg} \times .5\text{kg}}_{\text{sugar cost}} + \underbrace{\$3/\text{kg} \times .2\text{kg}}_{\text{choc cost}}$

12

$$\begin{array}{r} \\ \text{S} \\ \text{C} \end{array} \begin{array}{c} \text{cc} \quad \text{mm} \quad \text{ad} \\ \begin{pmatrix} & & \\ & & \end{pmatrix} \end{array} \text{NO} \begin{array}{r} \\ \text{SD} \\ \text{MC} \\ \text{Ma} \end{array} \begin{array}{c} \text{S} \quad \text{C} \\ \begin{pmatrix} & \\ & \\ & \end{pmatrix} \end{array}$$

Matrix multiplication and linear equations:

Example

Write the following system of linear equations as a matrix equation

$2x - 3y = 6$ row 1

$-x + 2y = 4$

Answer

$$\begin{array}{ccc} A & X & B \\ \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} & \begin{bmatrix} x \\ y \end{bmatrix} = & \begin{bmatrix} 6 \\ 4 \end{bmatrix} \end{array}$$

$(2\times 2)\,(2\times 1) = (2\times 1)$

$$\Rightarrow \begin{bmatrix} 2x + (-3)(y) \\ (-1)(x) + 2y \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

$$AX = B$$

$\frac{ax}{a}=\frac{b}{a} \Rightarrow x=b/a$

2.6 Inverse of a Square Matrix

$AX=B$

$1\frac{1}{2}$ or $2^{-1} = .5$

For any non-zero real number r, the reciprocal (or inverse) is $\frac{1}{r}$ or r^{-1}

$2\overline{)1}$

Multiplicative identity:

$$r \cdot r^{-1} = 1 \Rightarrow 2 \cdot \frac{1}{2} = 2 \cdot 2^{-1} = 1$$

For matrices, the inverse is A^{-1} and it is defined by

$$A \cdot A^{-1} = I = A^{-1}A$$

A matrix with no inverse is called ***singular***.

If needed, find the inverse with the x^{-1} function on the calculator.

The one use of matrix inverses is to solve matrix equations.

Solve the matrix equation AX = B for X

$$AX = B$$
$$\underbrace{A^{-1}A}X = A^{-1}B$$
$$IX = A^{-1}B$$
$$X = A^{-1}B$$

$$ax = b$$
$$\frac{\cancel{a}x}{\cancel{a}} = \frac{b}{a}$$
$$1x = b/a$$
$$x = b/a$$

$AXA^{-1} \Rightarrow$ A Mess

serif

1 sans serif

$|$?

Solve the matrix equation D = X – AX for X.

$$D = X - AX$$
$$= IX - AX$$
$$D = (I - A)X$$
$$(I-A)^{-1}D = (I-A)^{-1}\underline{(I-A)X}$$
$$(I-A)^{-1}D = IX$$
$$\boxed{X = (I-A)^{-1}D}$$

$$d = x - ax$$
$$= (1-a)x$$
$$\frac{d}{(1-a)} = \frac{(1-a)}{(1-a)}x$$

$$(\text{mm})^{-1}(\text{mm}) = I$$

$$D = X - XA = XI - XA = X(I-A)$$
$$D(I-A)^{-1} = X(I-A)(I-A)^{-1} = XI$$
$$X = D(I-A)^{-1}$$

Matrix inverses can be use to encrypt messages.

First, assign each letter of the alphabet a number:

1 to A	2 to B	3 to C	4 to D	5 to E	6 to F	7 to G
8 to H	9 to I	10 to J	11 to K	12 to L	13 to M	14 to N
15 to O	16 to P	17 to Q	18 to R	19 to S	20 to T	21 to U
22 to V	23 to W	24 to X	25 to Y	26 to Z	27 to space	

So the word aggies would be written

1 7 7 9 5 19

To make this more difficult to decode, we can put the letters in a message matrix. Our encoding matrix will be 3x3, so our message will need to have ~~3 rows:~~

$$M = \begin{bmatrix} 1 & 7 \\ 7 & 9 \\ 5 & 19 \end{bmatrix}$$

And multiply by an encoding matrix $E = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 1 & 4 \end{bmatrix}$ (any square matrix that's not singular and has pos intgrs)

longer

$(3\times3)\,(3\times2) = (3\times2)$

$$EM = \begin{bmatrix} 30 & 82 \\ 69 & 187 \\ 29 & 99 \end{bmatrix}$$

Send off 30 82 69 187 29 99

$\Downarrow$

STO [C]

Decode the message,

coded msg

$$M = E^{-1}\,(EM) = E^{-1}\begin{bmatrix} 30 & 82 \\ 69 & 187 \\ 29 & 99 \end{bmatrix} = \begin{bmatrix} 1 & 7 \\ 7 & 9 \\ 5 & 19 \end{bmatrix}$$

4 cols

3 rows

Decode the message below using the encryption matrix E.

150 114 149 178 113 184 182 148 227 260 193 269

35 82 100 143 98 175 259 329

52 95 115 151