SENSITIVITY ANALYSIS

A company produces two models of hibachis, model A and model B. To produce each model A requires 3 lbs of cast iron and 6 minutes of labor. Each model B requires 4 lbs of cast iron and 3 minutes of labor. The profit for each model A is $2 and the profit for each model B is $1.50. If 1000 lbs of cast iron and 20 hours of labor are available each day, how many hibachis of each model should be produced to maximize profits?

x = # of model A hibachis, y = # of model B hibachis, and P = profit in $
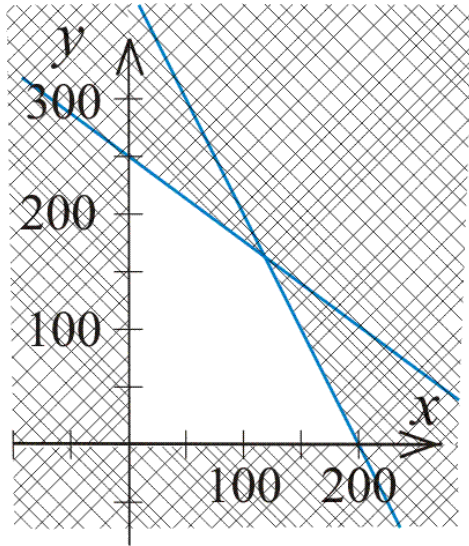
Maximize $P = 2x + 1.5y$

Subject to

$3x + 4y \leq 1000$ (pounds of iron)

$6x + 3y = 1200$ (minutes of labor)

$x, y \geq 0$

	$P = 2x + 1.5y$	
(0, 0)	0	
(0, 250)	375	
(120, 160)	480	maximum
(200, 0)	400	

But, when businesses are looking to maximize profits, it is possible to change values in the initial problem. Perhaps they will buy more or less iron. Hire or fire a worker. Change the price (and therefore the profit) on an item. We will look at how changes affect the bottom line in this section on sensitivity analysis.

Changes in the Objective Function

Change the profit on Model A hibachis to a (rather than 2) so $P = ax + 1.5y$ The slope of the old profit equation was

$$P = 2x + 1.5y \rightarrow y = -\left(\frac{4}{3}\right)x + \left(\frac{2}{3}\right)P$$

Where the line is depends on how big P is, but the slope is always

$$m_O = -\left(\frac{4}{3}\right) \text{ (the slope of the objective function).}$$

The new objective function will have a different slope, $P = ax + 1.5y \rightarrow$

$$y = -\left(\frac{2}{3}\right)ax + \left(\frac{2}{3}\right)P \text{ so } m_O = -\left(\frac{2}{3}\right)a$$

What are the slopes of the constraints?

Labor: $6x + 3y = 1200 \rightarrow y = \frac{1}{3}(-6x + 1200)$ or $m_L = -2$

Iron: $3x + 4y = 1000 \rightarrow y = \frac{1}{4}(-3x + 1000)$ or $m_I = -0.75$

As long as the slope of the objective function is between these values, we will end up at the same maximum. Solve for the possible values of a:

$$m_L < m_O < m_I \rightarrow -2 < -\left(\frac{2}{3}\right)a < -0.75 \text{ or } 1.125 < a < 3$$

So for profits between $1.125 and $3 on each of the Model A hibachis the maximum profit will occur at the same spot, 120 model A's and 160 model B's. Of course HOW MUCH profit will go up or down from $480, depending on a.

If a=1.125, then the slope of the profit equation will be

$$-\left(\frac{2}{3}\right)1.125 = -0.75 = m_I$$

The objective function will be parallel to the iron equation, so there will be a parametric solution.

Values of a smaller than $a = 1.125$ means the location of the maximum will shift to the point (250, 0) and it is not profitable to make any Model A's at all.

If $a = 3$, then the slope will be

$$-\left(\frac{2}{3}\right)3 = -2 = m_L$$

The objective function will be parallel to the labor equation. If the profit on model A is larger than 3, then the location of the maximum profit will move to (200, 0) and only model A's will be made.

A similar analysis will give you a range of values for the profit on B that will leave the location of the maximum unchanged.

If a resource is left over at the maximum, then that constraint is non-binding. A binding constraint means that the resource fully used at the maximum.

To find if anything is "leftover", you can put the x and y values at the maximum into the constraints and see if they are fully used or not. On the graph

Changes in the Resources

Say we have h pounds more or less of iron than our original 1000 pounds. Then the constraint will be

$$3x + 4y = 1000 + h \;\rightarrow\; y = -0.75x + 250 + \frac{h}{4}$$

Slope is unchanged from original problem, but y-intercept is now 250+h. The intersection of this line and the labor line ($6x + 3y = 1200$ or $y = -2x + 400$) will also shift.

$$-2x + 400 = -0.75x + 250 + \frac{h}{4} \;\rightarrow\; x = 120 - 0.2h \text{ and } y = 160 + 0.4h$$

We can't ever let x be negative so $x = 120 - 0.2h \geq 0 \;\rightarrow\; h \leq 600$

Where is this? If $h = 600$ then we had 1600 pounds of iron and the line only just touches the feasible region. If we had more than 1600 pounds of iron, the iron constraint would become non-binding.

In addition, y can't be negative either, so $y = 160 + 0.4h \geq 0 \;\rightarrow\; h \geq -400$

If $h = -400$, then we have 600 pounds of iron and the iron line and the labor line would just meet at the x axis. For the iron constraint to be meaningful, it must be in the range $-400 \leq h \leq 600$.

Shadow Prices

When we have 1000+h pounds of iron, our solution is at $(120 - 0.2h, 160 + 0.4h)$ and

$$P = 2x + 1.5y = 2(120 - 0.2h) + 1.5(160 + 0.5h) = 480 + 0.2h$$

This can be read as a $0.20 increase in profits for every 1 pound more of iron that is available. The shadow price for iron is $0.20.