

Chapter 6: SETS AND COUNTING**6.1 Sets and Set Operations**

A *set* is a collection of objects.

The objects in a set are the *elements* or *members* of the set.

→ Always enclose the elements of a set in curly brackets.

A set with the numbers $-1, 1, 0$ would be written as $\{-1, 1, 0\}$

$1 \in \{-1, 1, 0\}$ where \in is read "is an element of"

Define $S = \{-1, 1, 0\}$ then $1 \in S$

More notation:

- 0 is the symbol for the real number zero
- $\{0\}$ is a set with one element, the real number zero
- \emptyset is a set with zero elements, the empty set. Alternative is $\{\}$.
- $\{\emptyset\}$ is a set with one element, the symbol for the empty set.

Two sets are *equal* ($=$) if they contain exactly the same elements (order doesn't matter).

$$\{1, 2, 3\} = \{3, 2, 1\}$$

They are *not equal* (\neq) if they don't contain the same elements.

$$\{1, 2, 3\} \neq \{2, 3\}$$

Set builder notation: Describe the set in terms of its properties,

$$A = \{x \mid x \text{ is a positive even integer less than } 17\}$$

Roster notation: List the elements of the set.

$$A = \{2, 4, 6, 8, 10, 12, 14, 16\}$$

$$= \{2, 4, 6, \dots, 16\}$$

Subset: Set B is a subset of set A (written $B \subseteq A$) if every element in B is in A .

$$B = \{x \mid x \text{ is a positive multiple of } 4 \text{ less than } 17\}$$

$$B \subseteq A \quad A \subseteq A$$

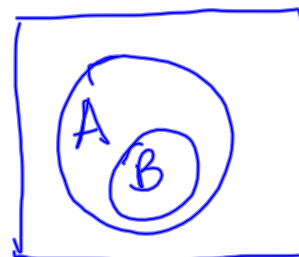
Proper Subset: Set B is a proper subset of set A (written $B \subset A$) if $B \subseteq A$ and $A \neq B$.

Universal set: The set from which all the member of other sets will be drawn. Called U .

$$U = \{x \mid x \text{ is a positive integer less than } 17\}$$

Venn Diagram notation:

- A rectangle represents the universal set
- Circles are sets in the universal set.



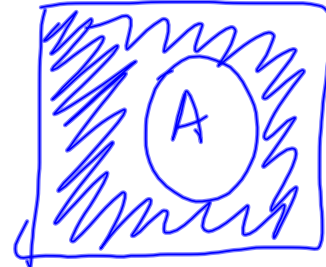
Example

Show the relationship between A and B (defined above) in a Venn diagram.

Given a set A and a universal set U , the elements that are in U and are NOT in A is called the **complement** of A or A^c .

$$A^c = \{x \mid x \in U \text{ and } x \notin A\}$$

shade A^c on the Venn diagram



Example

From the last example, A is the set of even integers, what is A^c in roster notation?

$$A^c = \{1, 3, 5, \dots, 15\}$$

Those elements that belong to both A ^{A (and)} B are in the **intersection** of A and B , $A \cap B$.

$$\begin{aligned} A \cap B &= \{x \mid x \in A \text{ and } x \in B\} \\ &= \{x \in U \mid x \in A \text{ and } x \in B\} \end{aligned}$$



Example

Let $U = \{x \mid x \text{ is a card in a standard deck of 52 playing cards}\}$

$R = \{x \mid x \text{ is a red card}\}$

$Q = \{x \mid x \text{ is a queen}\}$

Find $R \cap Q$ in roster notation.

$$R \cap Q = \{QH, QD\} = \{Q\heartsuit, Q\diamondsuit\}$$

If two sets have no elements in common, that is $A \cap B = \emptyset$, then the sets are *disjoint*.

Example

If $K = \{x \mid x \text{ is a king}\}$, find $K \cap Q$ in roster notation.

$$K \cap Q = \emptyset \text{ or } \{ \}$$

Those elements that belong to A *or* B are in the *union*, $A \cup B$.

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$



Note: this is the *inclusive or*, not the exclusive or

Example

Let $U = \{x \mid x \text{ is a card in a standard deck of 52 playing cards}\}$

$H = \{x \mid x \text{ is a heart card}\}$

$Q = \{x \mid x \text{ is a queen}\}$

Find $H \cup Q$ in roster notation.

$$H \cup Q = \{AH, 2H, 3H, \dots, QH, KH, QS, QD, QC\}$$

Example

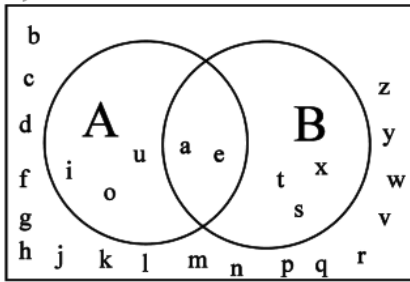
Let $U = \{x|x \text{ is a letter in the English alphabet}\} = \{a, b, c, \dots, z\}$

$A = \{x|x \text{ is a vowel}\} = \{a, e, i, o, u\}$

$B = \{x|x \text{ is a letter in the word texas}\} = \{t, e, x, a, s\}$

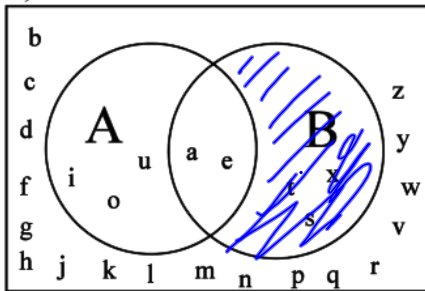
Find the following sets in roster notation.

a) What is $A \cap B$?



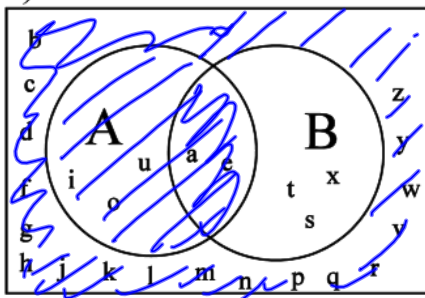
$\{a, e\}$

b) What is $A^c \cap B$?



$\{t, x, s\}$

c) What is $A \cup B^c$?



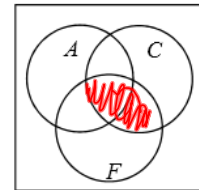
$\{a, b, c, \dots, r, u, v, w, y, z\}$

Example

People at a home show were surveyed to see if they planned on replacing their kitchen countertops (C), their kitchen floor (F) or their kitchen appliances (A). Shade the following regions on the Venn diagram and express the region in set notation. $\Rightarrow A, C, F \cup A, C, F$

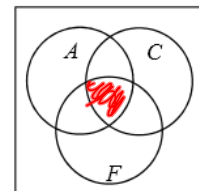
People who planned to replace their countertops and floor

$$C \cap F$$



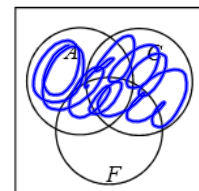
People who planned to remodel all three features

$$A \cap C \cap F$$



People who planned to replace their appliances or countertops

$$A \cup C$$



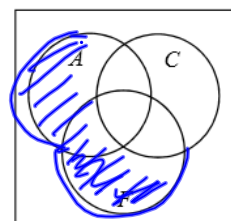
People who planned to replace exactly one of these features

$$(A \cap C^c \cap F^c) \cup (A^c \cap C \cap F^c) \cup (A^c \cap C^c \cap F)$$



People who planned to replace their kitchen appliances or floor, but not their countertops.

$$(A \cup F) \cap C^c$$

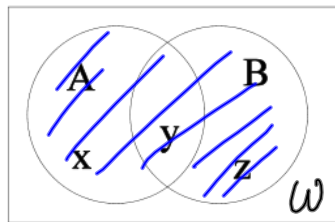


6.2 The Number of Elements in a Finite Set

The number of elements in set A is $n(A)$.

if $A = \{x|x \text{ is a letter in the English alphabet}\}$, then $n(A)=26$.

If $A = \emptyset$ then $n(A) = 0$.



$$n(A \cup B) = x + y + z = (x + y) + (y + z) - y$$

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

UNION RULE

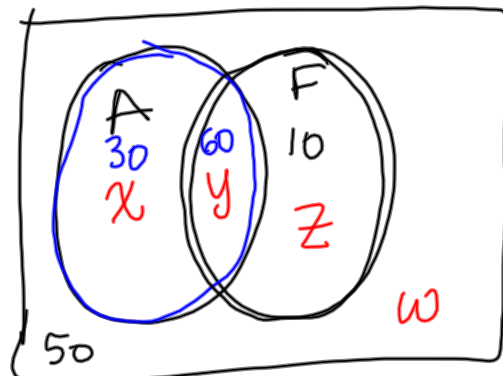
Example $A = \{x|x \text{ is an AM radio}\}$

A store has 150 clocks in stock. 100 of these clocks have AM or FM radios. 70 clocks had FM circuitry and 90 had AM circuitry.

a) How many had both AM and FM? 60

b) How many were AM only? 30

c) How many were FM only?



$$n(W) = 150 = x + y + z + w$$

$$n(A \cup F) = 100 = x + y + z$$

$$n(A) = 90 = x + y$$

$$n(F) = 70 = y + z$$

$$100 = 90 + 70 - n(A \cap F)$$

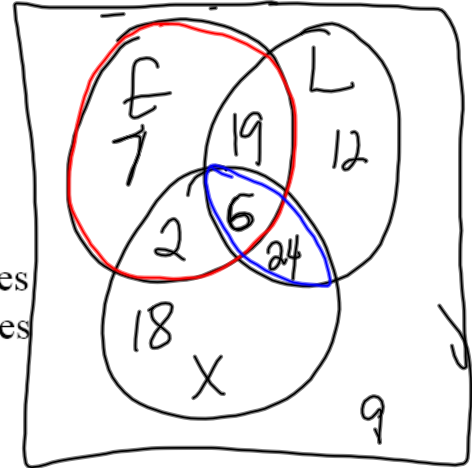
$$100 = 160 - n(A \cap F)$$

$$n(A \cap F) = 160 - 100 = 60$$

Example

We are given the following data about the contents of some delivery trucks,

- 34 trucks carried early peaches
- 61 trucks carried late peaches
- 50 trucks carried extra late peaches
- 25 trucks carried early and late peaches
- 30 trucks carried late and extra late peaches
- 8 trucks carried early and extra late peaches
- 6 trucks carried all three kinds of peaches
- 9 trucks carried no peaches



Display this information in a Venn diagram.

How many carried only late peaches? $n(E^c \cap L \cap X^c) = 12$

How many carried only one kind of peaches? $7 + 12 + 18 = 37$

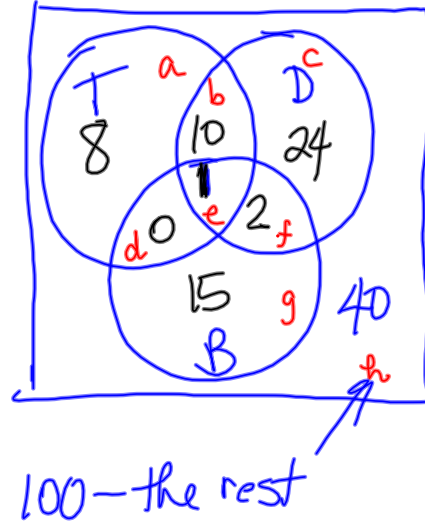
How many trucks went out? 97

Example

One hundred shoppers are interviewed about the contents of their bags and the following results are found:

- ~~19~~ bought Twinkies
- ~~37~~ bought diet soda
- 18 bought broccoli
- ~~1~~ bought broccoli, diet soda and Twinkies
- ~~11~~ bought Twinkies and diet soda
- ~~0~~ bought only Twinkies and broccoli
- ~~2~~ bought only diet soda

Display this information in a Venn diagram.



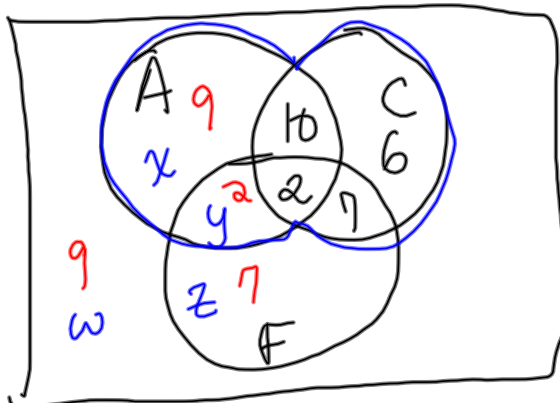
Example

Fifty-two people at a home show were surveyed to see if they planned on replacing their kitchen countertops (C), their kitchen floor (F) or their kitchen appliances (A). The following results were found:

- 22 people planned to replace exactly one of these features
- 18 people were planning to replace their kitchen appliances or floor, but not their countertops.
- ~~1~~ people planned to only replace their kitchen countertops
- ~~2~~ people planned to remodel all three features.
- ~~2~~ people planned to replace their countertops.
- 36 people planned to replace their appliances or countertops
- ~~2~~ people planned to replace their countertops and floor.

• 52 people in all

Display this information in a Venn diagram.



$$\begin{aligned}
 22 &= x + 6 + z \\
 18 &= x + y + z \\
 36 &= x + 10 + 6 + y + 2 + 7 \\
 52 &= x + 10 + 6 + y + 2 + 7 \\
 &\quad + w + z
 \end{aligned}$$

31 children were asked about their lunch

- 12 like CB
- 14 like PZ
- 9 like BU
- 5 like CB and PZ
- 4 like CB and BU
- 8 like PZ and BU
- 10 like none

How many like all 3?

6.3 The Multiplication Principle

Example

A fair coin is flipped three times. How many different outcomes are there for this experiment?

$$\frac{2}{\text{coin 1}} \cdot \frac{2}{\text{coin 2}} \cdot \frac{2}{\text{coin 3}} = 8$$

Multiplication Principle:

Suppose a task T_1 can be done N_1 ways and a task T_2 can be done N_2 ways and so on until task T_k can be done N_k ways. Then the number of ways of performing the tasks T_1, T_2, \dots, T_k is given by the product $N_1 \times N_2 \times \dots \times N_k$

Factorials: $n! = 1 \times 2 \times 3 \times \dots \times (n-1) \times n$. Note that $0! = 1$

Example

How many different license plates are possible where

a) The first character is a letter, the next two characters are digits and the last three characters are letters?

$$\frac{26}{\text{letter}} \cdot \frac{10}{\text{digit}} \cdot \frac{10}{\text{digit}} \cdot \frac{26}{\text{letter}} \cdot \frac{26}{\text{letter}} \cdot \frac{26}{\text{letter}} = 45,697,600$$

b) The first two characters are letters, the next character is a digit, the next a letter and the last three are digits?

$$\frac{26}{L} \cdot \frac{26}{L} \cdot \frac{10}{\#} \cdot \frac{26}{L} \cdot \frac{10}{\#} \cdot \frac{10}{\#} \cdot \frac{10}{\#} = 175,760,000$$

c) No characters are duplicated in part (b)

$$\frac{26}{L} \cdot \frac{25}{L} \cdot \frac{10}{\#} \cdot \frac{24}{L} \cdot \frac{9}{\#} \cdot \frac{8}{\#} \cdot \frac{7}{\#} = 7,862,400$$

$$\underline{15} \cdot \underline{14} \cdot \underline{13} \cdot \underline{12} \cdot \dots \cdot \underline{3} \cdot \underline{2} \cdot \underline{1} = 15!$$

Example

You have a group of 15 different books. Five are math books, four are chemistry and six are history books. How many different arrangements are possible if books of the same type are kept together?

$$\underbrace{3 \cdot 2 \cdot 1}_{\text{arr the groups}} \cdot \frac{5!}{M} \cdot \frac{4!}{C} \cdot \frac{6!}{H} = 12,441,600$$

MCH
 MHC
 CHM
 CMH
 HCM
 HMC

Example

A sales person has 6 prospects. How many ways can she arrange her schedule to see all 6?

$$\frac{6}{1^{\text{st}}} \cdot \frac{5}{2^{\text{nd}}} \cdot \frac{4}{3^{\text{rd}}} \cdot 3 \cdot 2 \cdot 1 = 6!$$

Example

How many different 4 digit numbers can be made from the digits

1, 2, 3, 4, 5, 6, 7

a) If there are no restrictions? $\frac{7}{\quad} \cdot \frac{7}{\quad} \cdot \frac{7}{\quad} \cdot \frac{7}{\quad} = 2401$

b) If the number must be even? $\frac{7}{\quad} \cdot \frac{7}{\quad} \cdot \frac{7}{\quad} \cdot \frac{3}{\text{even}}$

c) If it is even and there are no repeats? $\frac{6}{\quad} \cdot \frac{5}{\quad} \cdot \frac{4}{\quad} \cdot \frac{3}{\text{even}}$

d) If four of the same digit is not allowed?

$$2401 - 7 = 2394$$

6.4 Combinations and Permutations

Example

How many ways can 10 students be seated in a row of 10 chairs?

$$\underline{10} \cdot \underline{9} \cdot \underline{8} \cdots \underline{3} \cdot \underline{2} \cdot \underline{1} = 10! \quad P(10,10)$$

Example

How many ways can 4 of 10 students be seated in a row of 4 chairs?

$$\underline{10} \cdot \underline{9} \cdot \underline{8} \cdot \underline{7} = 5040 \quad P(10,4)$$

10 MATH PRB nPr 4

Permutations: If we have a finite set of n elements and we want to place r of them in an arrangement, we say the number of permutations of n things arranged r at a time is $P(n, r)$.

Example

How many ways can gold, silver and bronze medals be awarded in a race of 12 people?

$$\frac{12}{G} \cdot \frac{11}{S} \cdot \frac{10}{B} = 1320 = P(12, 3)$$

Example

How many ways can a group of 4 students be chosen from 10 students?

$$\frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} = 210 = 10 \text{ math Prob } nCr 4 = C(10, 4)$$

Combinations: If we have a finite set of n elements and we want to take r of them in an group, we say the number of combinations of n things grouped r at a time is $C(n, r)$.

Example

How many ways can a hand of 6 clubs be chosen from a standard deck?

$$\frac{C(13,6)}{13 \text{ MAT} + \text{PRB } nCr 6} = 1716 \quad \Bigg\| \text{ Prob} = \frac{C(13,6)C(39,0)}{C(52,6)} \text{ (later)}$$

Example

From a group of 12 people, how many ways can a committee of 4 be formed if one person is the chair of the committee?

$$\frac{12}{\text{Chair}} \cdot \frac{C(11,3)}{\text{rest}} = 1980 = \frac{C(12,4)}{\text{Committee}} \cdot \frac{4}{\text{Chair}}$$

Example

A bag contains 6 blue, 1 green and 3 pink jelly beans. You choose 3 at random. How many samples are possible in which

a) the jelly beans are all blue?

$$\frac{C(6,3)}{3 \text{ Blue}} \frac{C(1,0)}{0 \text{ Green}} \frac{C(3,0)}{0 \text{ Pink}} = 20$$

b) the jelly beans are all green?

0

c) the jelly beans are all pink? $C(3,3)$

1

d) there are 2 blue and 1 pink?

$$\frac{C(6,2)}{\text{Blue}} \frac{C(1,0)}{G} \frac{C(3,1)}{P} = 15 \times 1 \times 3 = 45$$

e) How many ways to choose 3 jelly beans?

$$C(10,3) = 120$$

f) How many ways to choose no blue?

$$\frac{C(6,0)}{\text{Blue}} \frac{C(4,3)}{\text{not Blue}} = 4$$

g) How many ways to choose at least one blue?

$$1B 2B^c + 2B 1B^c + 3B 0B^c$$

$$C(6,1)C(4,2) + C(6,2)C(4,1) + C(6,3)C(4,0)$$

$$36 + 60 + 20 = 116$$

