

## Introduction to Game Theory

### ***G.1 Decision Making***

Game theory is a mathematical model that provides a systematic way to deal with decisions under circumstances where the alternatives are ambiguous or unclear. Game theory is a relatively recent mathematical tool developed to help make decision in business, warfare and social situations. It can also be applied to recreational games as well.

### **Payoff Matrices**

*Example:* A farmer is considering installing irrigation equipment. If he installs the system and it is a dry year, he will make a profit of \$50,000. If he installs the system and it is a wet year, he will have a profit of \$10,000. If he does not install the system and it is a dry year, he will lose \$20,000. If he does not install the system and it is a wet year, he will make \$35,000. *profit*

What should the farmer do? First we must define the problem mathematically. In this case there are two *states of nature*. The states of nature are things that you cannot control; namely, a wet or dry year. There are two *strategies*. Strategies are things that you can control; namely install the irrigation system or not.

The consequence of each course of action (strategy) under each state of nature is called a *payoff*. We will organize these payoffs in a matrix, called the *payoff matrix*. We typically put the different strategies on the rows and the states of nature on the columns.

		<i>States of nature</i>	
		<i>dry year</i>	<i>wet year</i>
<i>Strategies</i>	<i>system</i>	(	50,000    10,000)
	<i>no system</i>	(	-20,000    35,000)

What will the farmer do if he is an *optimist*?

	<i>dryyear</i>	<i>wetyear</i>
<i>system</i>	<u>50,000</u>	10,000
<i>nosystem</i>	-20,000	<u>35,000</u>

Install the system  
expecting a dry year

What will the farmer do if he is a *pessimist*?

	<i>dryyear</i>	<i>wetyear</i>
<i>system</i>	50,000	<u>10,000</u>
<i>nosystem</i>	<u>-20,000</u>	35,000

Install the system  
expecting a wet year

What will the farmer do if his almanac says there is a 40% chance that it will be a dry year?

$$\begin{aligned} \text{system: } & (.4)(50000) + (.6)(10000) = \$26,000 \quad \star \\ \text{nosystem: } & (.4)(-20000) + (.6)(35000) = \$13,000 \end{aligned}$$

What if the probability it is a dry year is 10%?

$$\begin{aligned} \text{system: } & (.1)(50000) + (.9)(10000) = \$14,000 \\ \text{nosystem: } & (.1)(-20000) + (.9)(35000) = \$29,500 \quad \star \end{aligned}$$

## Strictly Determined Games

Game theory can be applied to a game that two people play against each other. In the previous section we had a player against nature, but now we will have two players. We will need some definitions for a game played with two players.

We will call the players A and B and the game is called a two-person game.

Player A controls the ROW decisions

Player B controls the COLUMN decisions.

What the player chooses is called their strategy.

The convention is that a positive number in the payoff matrix means that B pays A. A negative number means A pays B.

*Example:* Consider the payoff matrix below. What are the possible outcomes?

$$A \begin{matrix} \begin{matrix} \text{H} & \text{T} \end{matrix} \\ \begin{pmatrix} 3 & -1 \\ -4 & 7 \end{pmatrix} \end{matrix}$$

A picks R1, B picks C1 payoff is 3 to A

A picks R1, B picks C2 payoff is 1 to B

A picks R2, B picks C1, payoff is 4 to B

A picks R2, B picks C2, payoff is 7 to A

This kind of game where if one person loses the other wins that amount is called a **zero-sum game**. Some games (like a model of the stock market) are not zero-sum games.

**Dominated Strategies**

Consider the following payoff matrix:

$$\begin{pmatrix} 1 & -1 & 3 \\ -2 & 0 & 5 \\ 2 & 4 & 6 \end{pmatrix}$$

First look at the matrix from A's point of view....

If one row is ALWAYS a better option than another row, we removed the row that is worse. This is called *removing dominated strategies* – we remove a strategy if it would never be chosen anyway.

$$\begin{pmatrix} \cancel{1} & \cancel{-1} & \cancel{3} \\ \cancel{-2} & \cancel{0} & \cancel{5} \\ 2 & 4 & 6 \end{pmatrix} \Rightarrow (2 \quad \cancel{4} \quad \cancel{6}) \Rightarrow (2)$$

So we get a new payoff matrix with the dominated strategy removed:

Look at the new payoff matrix from B's view. B likes negative numbers, or at least small numbers.

Example: Remove any dominated strategies from the payoff matrix,

$\downarrow$     $\downarrow$    col 1 always better than col 2

$$\begin{pmatrix} -8 & \cancel{4} & -6 & 1 \\ -3 & \cancel{0} & -9 & 12 \end{pmatrix}$$

$\hookrightarrow \begin{pmatrix} -8 & -6 & \cancel{\phantom{0}} & \cancel{\phantom{1}} \\ -3 & -9 & \cancel{\phantom{0}} & \cancel{\phantom{1}} \end{pmatrix} \rightarrow \begin{pmatrix} -8 & -6 \\ -3 & -9 \end{pmatrix}$  done

The goal of game theory is to find the *optimum strategy* for each player. The optimum strategy will give the player the most payoff possible. The payoff that results from each player choosing their optimum strategy is called the *value* of the game.

The simplest strategy is for a player to always pick the same row (or same column) every time they play. This is called a *pure strategy*.

Consider the following payoff matrix:

$\downarrow$

$$A \begin{bmatrix} -5 & -6 & 4 \\ 1 & 2 & 3 \\ -1 & 3 & -2 \end{bmatrix}$$

Now look at each of A's strategies and see what B would choose as a response to each.

A chooses row	B chooses column	Payoff is
1	2	6 to B
2	1	1 to A
3	3	2 to B

A picks row 2 and wins at least!

$$\begin{bmatrix} -5 & \underline{-6} & 4 \\ \underline{1} & 2 & 3 \\ -1 & 3 & \underline{-2} \end{bmatrix}$$

underline smallest in each row  $\Rightarrow$  take the largest of these

Look at B's choices for columns and what A would do in response to each:

B chooses column	A chooses row	Payoff is
1	2	1 to A
2	3	3 to A
3	1	4 to A

B chooses col 1 and loses at most!

$$\begin{bmatrix} -5 & \underline{-6} & \textcircled{4} \\ \textcircled{1} & 2 & 3 \\ -1 & \textcircled{3} & \underline{-2} \end{bmatrix}$$

circle largest in each column  $\Rightarrow$  take the smallest of these

If a payoff matrix A has an entry  $a_{hk}$  that is the minimum in its row (row  $h$ ) and the maximum in its column (column  $k$ ), that entry  $a_{hk}$  is a **saddle point**.

If a game has a saddle point, then the optimum pure strategy for A is to choose the row with the saddle point and the optimum pure strategy for B is to choose the column with the saddle point. A game with a saddle point is a **strictly determined game**. A **fair game** has a value of 0.

A will pick row 2 and B coln 1 value is 1  
 favors player A

*Example* : Look at the payoff matrix below. Does it have a saddle point? Is it strictly determined? What is the value of the game? Is it fair? What are A and B's optimum strategies?

$$\begin{bmatrix} 1 & 4 & -3 & 1 & -1 \\ 2 & 5 & 0 & 4 & 10 \\ 1 & -3 & -2 & 5 & 2 \end{bmatrix}$$

Yes, a saddle point at Row 2 Col 3. Str. det b/c it has a saddle point. Value is 0 a fair game. A plays row 2 and B plays col 3.

**G.2 Mixes Strategies**

What if a game does not have a saddle point? There is no optimum pure strategy, but a player can have an optimum mixed strategy.

$$\begin{bmatrix} 6 & 2 \\ -1 & 10 \end{bmatrix}$$

Suppose that player A decides to choose row 1 and row 2 with equal probability. Next suppose that player B decides to choose column 1 two thirds of the time and column 2 one-third of the time. Finally, assume that A and B make their choices independently



Outcome	Probability	Payoff to A
R1, C1	$\frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$	6
R1, C2	$\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$	2
R2, C1	$\frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$	-1
R2, C2	$\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$	10

Expected value for A is

$$E = (6)\left(\frac{1}{3}\right) + (2)\left(\frac{1}{6}\right) + \left(\frac{1}{3}\right)(-1) + (10)\left(\frac{1}{6}\right) = 3\frac{2}{3} \text{ or } \$3.67$$

Say you play 10 games, A will be ahead about \$37

We can find the expected value of a generalized 2x2 game as shown:

$$M = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad A = [p_1 \ p_2]$$

$$B = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

$$E = p_1 \cdot q_1 \cdot a_{11} + p_1 \cdot q_2 \cdot a_{12} + p_2 \cdot q_1 \cdot a_{21} + p_2 \cdot q_2 \cdot a_{22}$$

$$A \cdot M \cdot B = E$$

*m choices for A*

If we have a larger game, one with  $m$  rows and  $n$  columns, then the payoff matrix will be an  $m \times n$  matrix.

A's strategy will be a  $1 \times m$  matrix

B's strategy will be an  $n \times 1$  matrix.

$$E = [p_1 \ p_2 \ \dots \ p_m] \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & & & \\ \vdots & & & \\ a_{m1} & & & a_{mn} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix}$$

$$= A M B$$



OPTIMUM MIXED STRATEGY.

What is the optimum mixed strategy for the payoff matrix  $\begin{bmatrix} p_1 & 6 & 2 \\ p_2 & -1 & 10 \end{bmatrix}$ ?

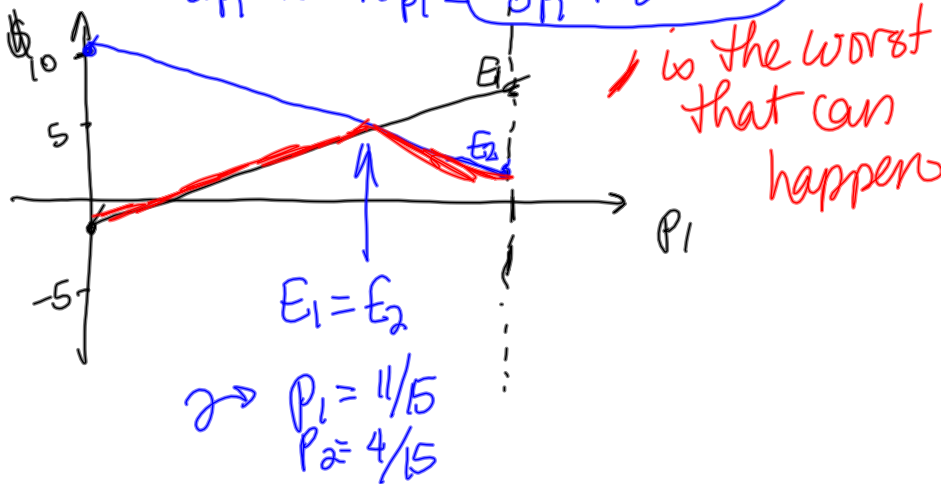
Assume A chooses row 1 with probability  $p_1$  and row 2 with probability  $p_2$ . Determine what A will win under the conditions that B always picks column 1 or B always picks column 2.

B picks col 1

$$E_1 = 6 \cdot p_1 + (-1)(1-p_1) = 6p_1 + (-1)(1-p_1) = 6p_1 - 1 + p_1 = 7p_1 - 1 = E_1$$

B picks col 2

$$E_2 = 2p_1 + 10p_2 = 2p_1 + 10(1-p_1) = 2p_1 + 10 - 10p_1 = -8p_1 + 10 = E_2$$



For B check out col 1 picked  $q_1$  and col 2 picked  $q_2 = 1 - q_1$ . See what happens for always pick row 1 and always pick rows

$$\Rightarrow q_1 = 8/15 \quad q_2 = 7/15$$

$$E = A M B = \begin{bmatrix} 6 & 2 \\ 15 \end{bmatrix} \text{ or } \$4.13$$

If we have a generalized payoff matrix,  $M = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

A's optimum mixed strategy is  $p_1 = \frac{a_{22} - a_{21}}{a_{11} + a_{22} - a_{12} - a_{21}}$  and  $p_2 = 1 - p_1$

B's optimum mixed strategy is  $q_1 = \frac{a_{22} - a_{12}}{a_{11} + a_{22} - a_{12} - a_{21}}$  and  $q_2 = 1 - q_1$

The number of cases of African flu has reached epidemic levels. The disease is known to have two strains with similar symptoms. Dr. Hyde has two medicines available. The first is 60% effective against the first strain and 40% effective against the second strain. The second medicine is completely effective against the second strain and completely ineffective against the first strain.

- Write a payoff matrix to show Dr. Hyde's results.
- What is Dr. Hyde's optimum strategy?
- What results can she expect?

(state of nature)

		strain 1	strain 2
strat	med 1	0.6	0.4
	med 2	0	1

$$b) = p_1 = \frac{1-0}{.6+1-0-.4} = \frac{1}{1.2} = \frac{5}{6}, p_2 = \frac{1}{6}$$

give med 1  $\frac{5}{6}$  of the time and med 2  $\frac{1}{6}$  of the time

$$c) q_1 = \frac{1-.4}{1.2} = \frac{.6}{1.2} = \frac{1}{2} \text{ so half have strain 1 and half have strain 2}$$

$$E = A \cdot M \cdot B = \frac{1}{2} \rightarrow \text{cures at least } \frac{1}{2}$$