1.2 Elementary Functions

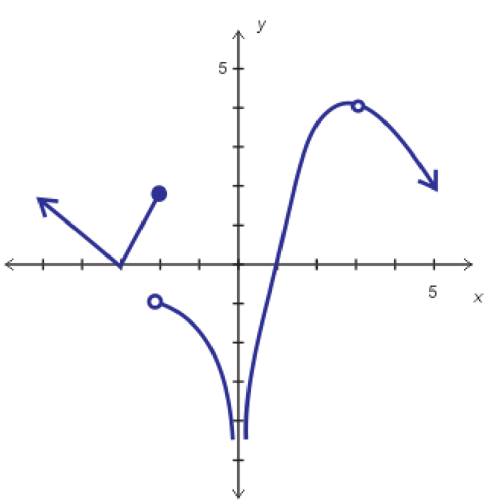
Let D and R be two nonempty sets. A **function** f from D to R is a rule that assigns to each element x in D one and only one element $y = f(x)$ in R.

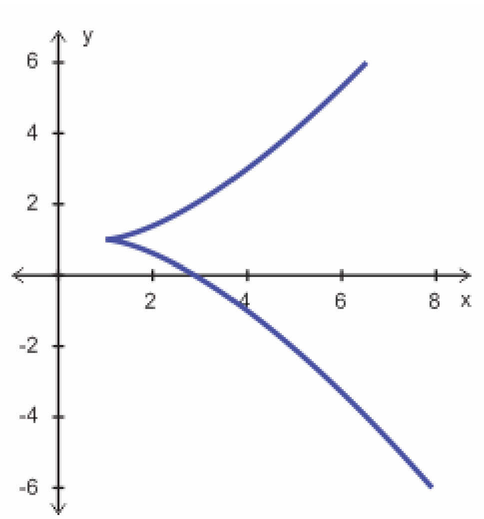
The set D in the definition is called the **domain** of f and it is the set of possible inputs for our function. The letter representing the elements in the domain is called the **independent variable**. In the definition above, the independent variable is x.

The set R is called the range of f and it is the set of all possible values of the output of our function. The letter representing the elements in the domain is called the **dependent variable**. In the definition above, the independent variable is y.

The **graph** of a function f consists of all points (x, y) such that x is in the domain of f and $y = f(x)$. A graph in the xy-plane represents a function if and only if every vertical line intersects the graph in at most one place. This is called the **vertical line test**.

Example: Which of the graphs below are graphs of functions?



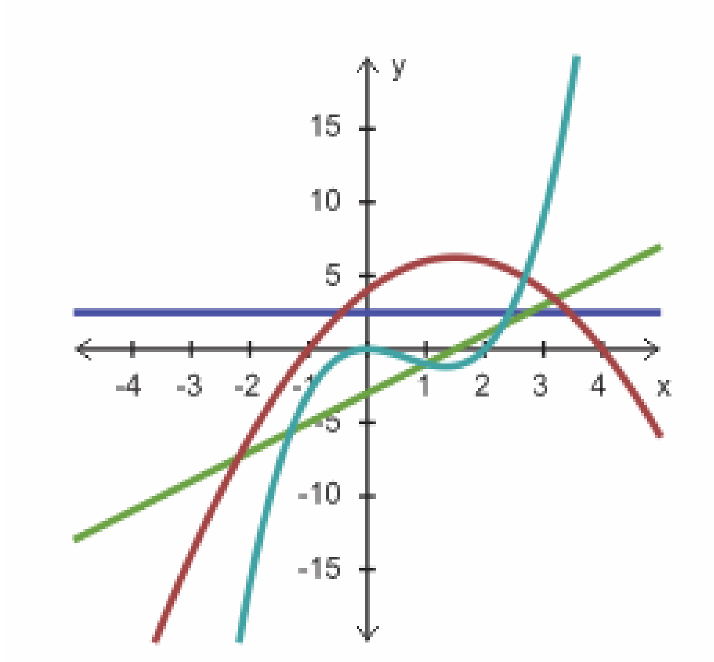


A **polynomial** is a function of the form

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

where n is a non-negative integer and a_i are real valued constants. The **leading coefficient** is a_n and $a_n \neq 0$. The **degree** of the polynomial is n. The domain of a polynomial is $\Re$ or $(-\infty, \infty)$.

Some common polynomials



degree 0: $y = 2.5$
constant function

degree 1: $y = 2x - 3$
linear function

degree 2: $y = x^3 - 2x^2$
quadratic function

degree 3: $y = -x^2 + 3x + 4$
cubic function

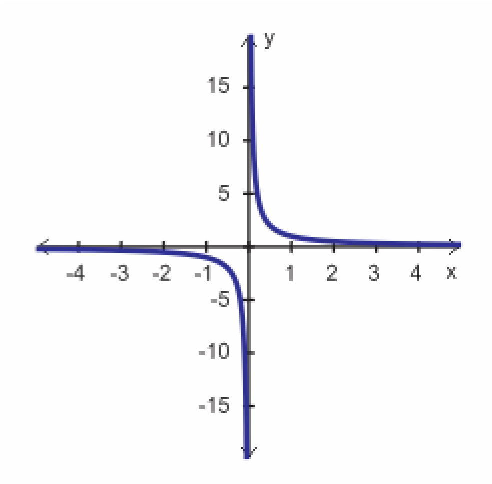
A rational function $f(x)$ is the quotient of two polynomial functions $p(x)$ and $q(x)$: $f(x) = \dfrac{p(x)}{q(x)}$

The domain of $f(x)$ is all real numbers for which $q(x) \neq 0$

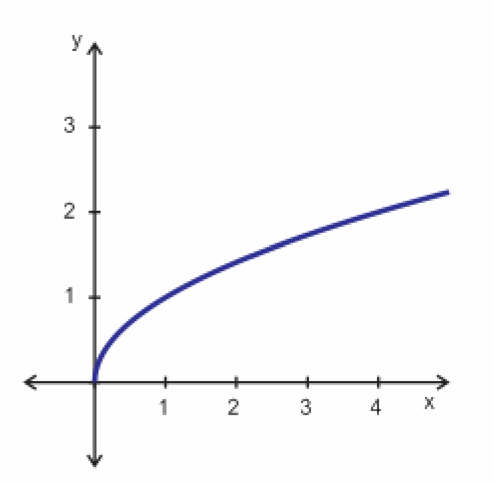
Example: What is the domain of $f(x) = \dfrac{x+1}{x^2 - 1}$?

A function of the form $f(x) = x^r$ where *r* is a real number is called a **power function**. Some examples of power functions:

$y = x^{-1}$



$y = x^{1/2}$



Example: In the case of $f(x) = \sqrt[n]{x}$ what is the domain?

$f(x) = a^x, a > 0, a \neq 1$ is called an **exponential function.** Exponential functions are often used to model growth and decay. If A_0 is the initial amount and *k* is the growth or decay rate, then the population at time *t* is given by

$$A(t) = A_0 e^{kt}$$

For exponential growth, $k > 0$ and for exponential decay, $k < 0$.

Example: A bacteria culture starts with 4000 bacteria and the population triples every 30 minutes. Find an expression for the number of bacteria after *t* hours.

The **half-life** of a substance is the amount of time it takes for half of the substance to disintegrate.

Example: After 3 days a sample of an unknown radioactive element is found to have decayed to 58% of its original amount. What is the half-life of this element? http://ie.lbl.gov/education/isotopes.htm

A function $f(x)$ is **one-to-one** provided that whenever $f(x_1) = f(x_2)$ then $x_1 = x_2$. One way to check if a function is one-to-one is to use the horizontal line test.

Let $f(x)$ be a one-to-one function with domain D and range R. Then the **inverse** function $f^{-1}(x)$ exists. The domain of $f^{-1}(x)$ is R and the range of $f^{-1}(x)$ is D. Moreover,

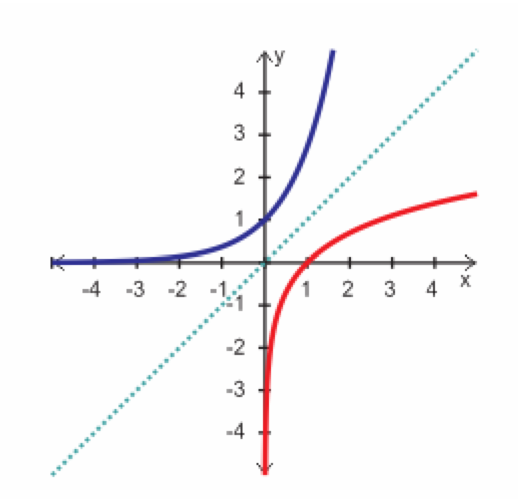
$$f(x) = y \quad \Leftrightarrow \quad f^{-1}(y) = x$$

Example: Find the inverse, and the domain and range of $f(x) = 5 - 4x^3$.

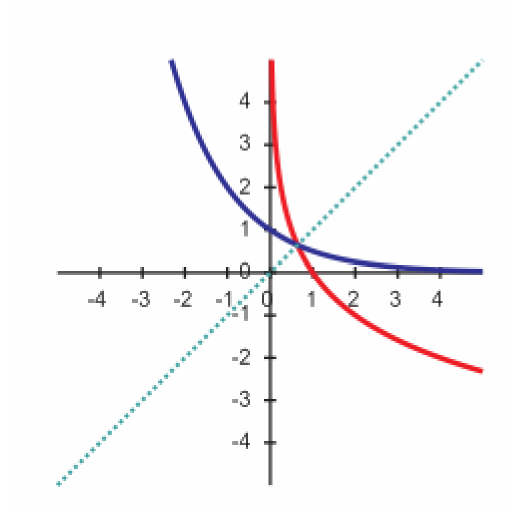
The inverse of the exponential function $f(x) = a^x$ is $f^{-1}(x) = \log_a x$.

Example: Find the domains and ranges of $f(x) = a^x$ and $f^{-1}(x) = \log_a x$

$y = e^x$ and $y = \ln x$



$y = \left(\frac{1}{2}\right)^x$ and $y = \log_{1/2} x$

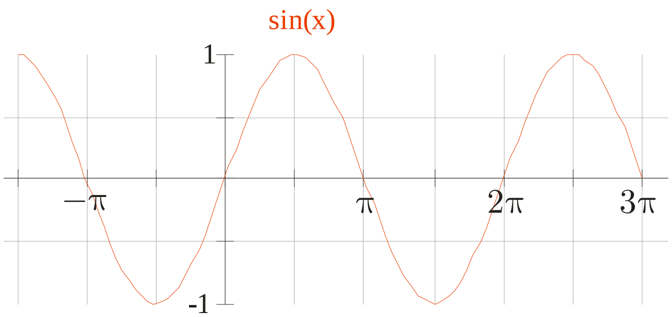


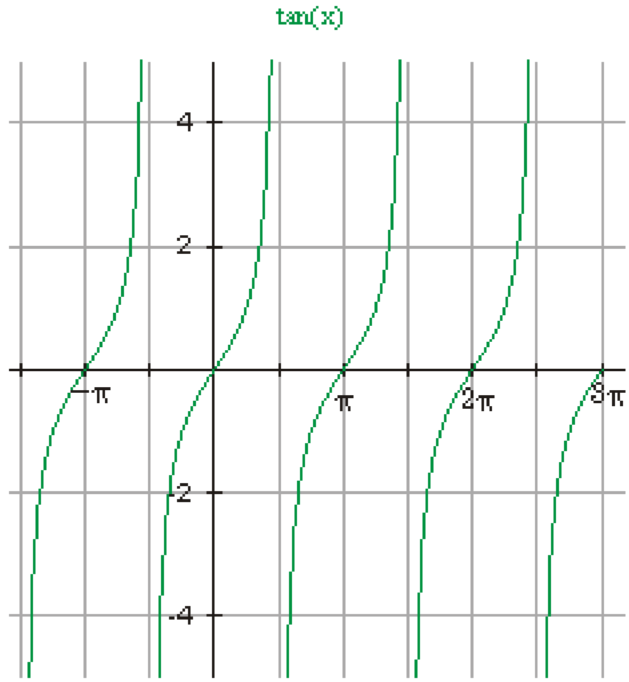
Example: Find the domain and range of each function

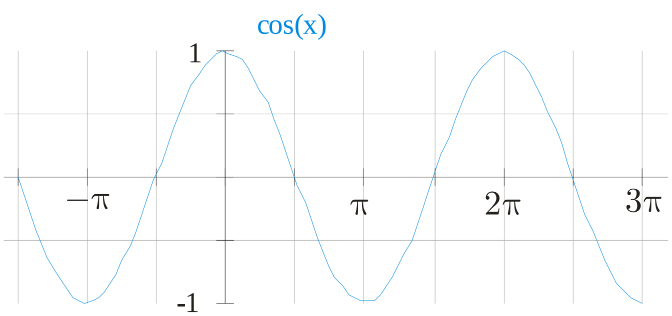
$f(x) = \log\left(x^2 - 3x + 2\right)$ $g(x) = \ln\left(x^3 - x\right)$

Example: Write the expressions 5^x and $\log_7(x+5)$ in terms of base *e*.

$f(x)$ is a **periodic** function if there is a positive constant *c* such that $f(x+p) = f(x)$. If *p* is the smallest number with this property, we call it the **period** of $f(x)$. The trigonometric functions are periodic functions.







For any real number *a* and $k \neq 0$, the functions $f(x) = a\sin(kx)$ and $g(x) = a\cos(kx)$ have an amplitude of $|a|$ and a period of $p = \dfrac{2\pi}{|k|}$.

Example: Find the amplitude and period of $f(x) = 3\sin(2x)$ and graph the function.