Clicker: Have you been to help session? (A) Yes (B) No

3.3 Limits at Infinity

Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x to be sufficiently large.

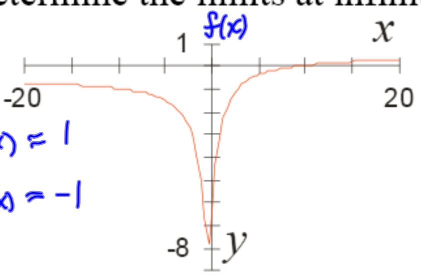
Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \to -\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x to be sufficiently large negative.

EXAMPLE 1

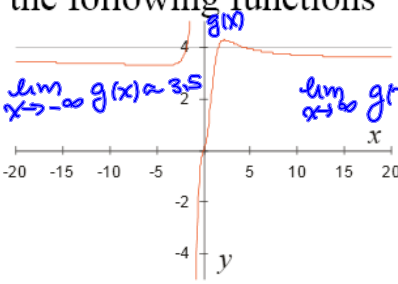
Determine the limits at infinity for the following functions



$\lim_{x \to \infty} f(x) = 1$

$\lim_{x \to -\infty} f(x) = -1$

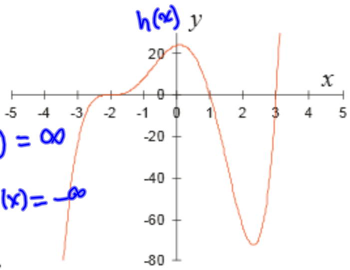
a)



$\lim_{x \to -\infty} g(x) \approx 3.5$

$\lim_{x \to \infty} g(x) \approx 3.5$

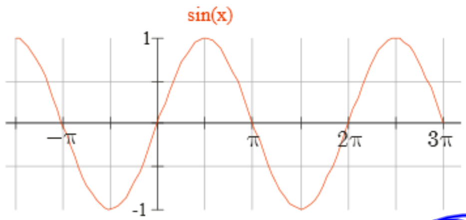
b)



$\lim_{x \to \infty} h(x) = \infty$

$\lim_{x \to -\infty} h(x) = -\infty$

c).



d)

Limit is (A) 0 (B) 1 (C) -1 (D) DNE (E) Other

If $r > 0$ is a rational number, then $\lim_{x \to \infty} \frac{1}{x^r} = 0$

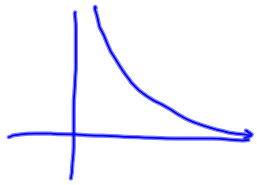
If $r > 0$ is a rational number such that x^r is defined for all x, then $\lim_{x \to -\infty} \frac{1}{x^r} = 0$

Example: Evaluate the following limits and justify each step

a) $\lim\limits_{x\to\infty}\dfrac{7x^3+4x}{2x^3-x^2+3}\dfrac{(1/x^3)}{(1/x^3)} = \lim\limits_{x\to\infty}\dfrac{7+4/x^2}{2-1/x+3/x^3} = \dfrac{\lim\limits_{x\to\infty} 7 + 4\lim\limits_{x\to\infty} 1/x^2}{\lim\limits_{x\to\infty} 2 - \lim\limits_{x\to\infty} 1/x + 3\lim\limits_{x\to\infty} 1/x^3}$

$= \dfrac{7+4(0)}{2-0+3(0)} = \dfrac{7}{2} = 3.5$

$\lim\limits_{x\to\infty}\dfrac{1}{x^r} = 0$



b) $\lim\limits_{x\to\infty}\dfrac{x^3-1}{x^4+1}\cdot\dfrac{1/x^4}{1/x^4} = \lim\limits_{x\to\infty}\dfrac{1/x-1/x^4}{1+1/x^4} = \dfrac{\lim\limits_{x\to\infty}\frac{1}{x} - \lim\limits_{x\to\infty} 1/x^4}{\lim\limits_{x\to\infty} 1 + \lim\limits_{x\to\infty} 1/x^4} = \dfrac{0-0}{1+0} = 0$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\frac{p(x)}{q(x)} = \begin{cases} 0 & \text{if } \deg(p) < \deg(q) \\ L \neq 0 & \text{if } \deg(p) = \deg(q) \\ \cancel{DNE}\ +/- \infty & \text{if } \deg(p) > \deg(q) \end{cases}$$

$\lim\limits_{x\to-\infty}\dfrac{1-x+5x^2}{3x-2x^2}\left(\dfrac{1/x^2}{1/x^2}\right) = \lim\limits_{x\to-\infty}\dfrac{1/x^2-1/x+5}{3/x-2} = \dfrac{0-0+5}{0-2}$

$3\lim\limits_{x\to\infty} 1/x$

$= -2.5$

Example: Find the following limits

a) $\lim_{x\to\infty}\sqrt{\frac{2x^2-1}{x+8x^2}} = \lim_{x\to\infty}\sqrt{\frac{2x^2-1}{x+8x^2}\frac{(1/x^2)}{(1/x^2)}} = \lim_{x\to\infty}\sqrt{\frac{2-1/x^2}{1/x+8}} = \sqrt{\frac{2}{8}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$

b) $\lim_{x\to-\infty}\frac{6t^2+5t}{(1-t)(2t-3)} = \lim_{x\to-\infty}\frac{6t^2}{-2t^2} = -3$

c) $\lim_{x\to-\infty}\frac{x^4+2x+3}{x(x^2-1)} = \lim_{x\to-\infty}\frac{x^4}{x^3} = \lim_{x\to-\infty} x = -\infty$

if $x=-10 \Rightarrow \sqrt{(-10)^2} = 10 \neq x$ so use $-x$

d) $\lim_{x\to-\infty}\frac{\sqrt{x^2+4x}}{4x+1} = \lim_{x\to-\infty}\frac{\sqrt{x^2}}{4x} = \lim_{x\to-\infty}\frac{-\cancel{x}}{4\cancel{x}} = -\frac{1}{4}$

e) $\lim_{x\to\infty}\left(\sqrt{x^2+3x+1}-x\right) * \frac{(\sqrt{x^2+3x+1}+x)}{(\sqrt{x^2+3x+1}+x)} = \lim_{x\to\infty}\left(\frac{\cancel{x^2}+3x+1-\cancel{x^2}}{\sqrt{x^2+3x+1}+x}\right)$

$= \lim_{x\to\infty}\frac{3x+1}{\sqrt{x^2+3x+1}+x} * \frac{1/x}{1/x} = \lim_{x\to\infty}\frac{3+1/x}{\sqrt{1+3/x+1/x^2}+1} = \frac{3+0}{\sqrt{1+0+0}+1} = \frac{3}{2}$

$\sqrt{1/x^2}$

The line $y = L$ is called a horizontal asymptote of the curve $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L$$

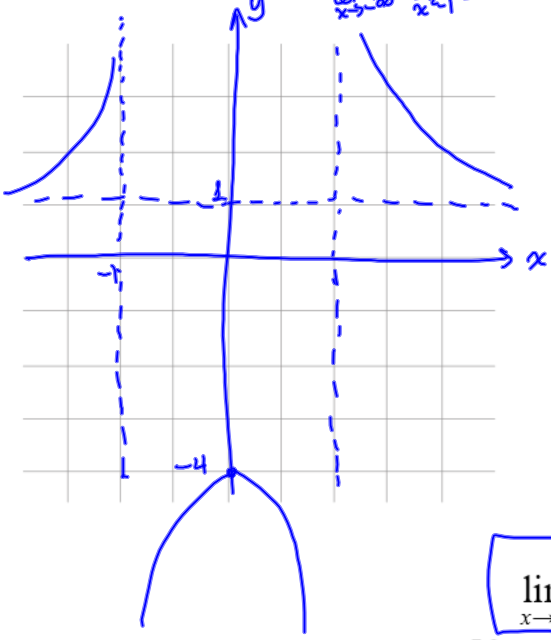
Example: Find the horizontal and vertical asymptotes of each curve.

a) $y = \dfrac{x^2+4}{x^2-1}$

VA @ $x = 1$, $x = -1$

HA: $\lim_{x\to\infty} \frac{x^2+4}{x^2-1} = 1$

$\lim_{x\to-\infty} \frac{x^2+4}{x^2-1} = 1$

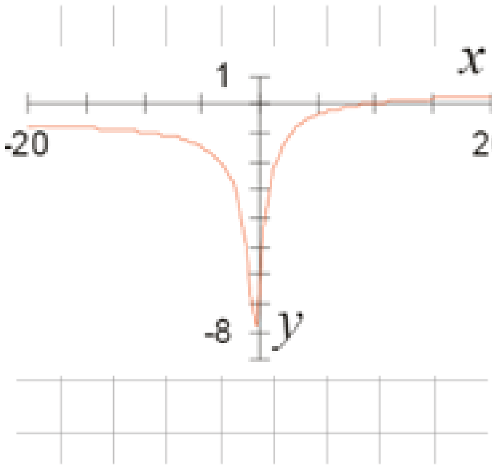


b) $y = \dfrac{x-9}{\sqrt{4x^2+3x+2}}$

NO VA

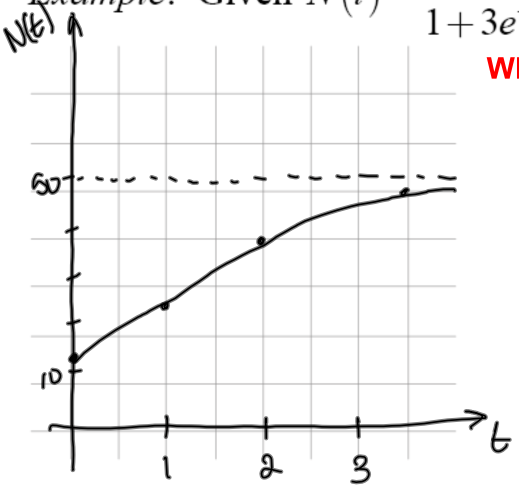
HA: $\lim_{x\to\infty} y = \frac{1}{2}$

$\lim_{x\to-\infty} y = -\frac{1}{2}$



$$\boxed{\lim_{x\to\infty} e^{-x} = 0}$$

Example: Given $N(t) = \dfrac{50}{1+3e^{-2t}}$, find $\lim_{t\to\infty} N(t)$ and graph $N(t)$



What is N(0)? 12.5 $= \dfrac{50}{1+3e^{-2(0)}} = \dfrac{50}{1+3}$

$$\lim_{t\to\infty} \frac{50}{1+3e^{-2t}} = \frac{\lim_{t\to\infty} 50}{\lim_{t\to\infty} 1 + 3\lim_{t\to\infty} e^{-2t}} = \frac{50}{1+3(0)}$$

$= 50$