3.4 The Sandwich Theorem and Some Trigonometric Limits

Sandwich Theorem: If $f(x) \le g(x) \le h(x)$ for all x in an open interval that contains a (except possibly at a) and

$$\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$$

then

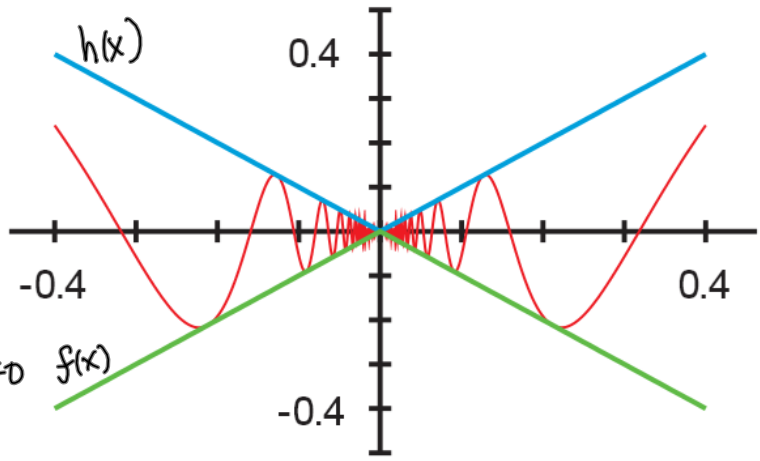
$$\lim_{x\to a} g(x) = L$$

Example: What is $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right)$?

$f(x) = -|x|$

$g(x) = x\sin\left(\frac{1}{x}\right)$

$h(x) = |x|$



Since $\lim_{x\to 0} |x| = 0$, $\lim_{x\to 0} -|x| = 0$

and $-|x| \le x\sin\left(\frac{1}{x}\right) \le |x|$

$\Rightarrow \lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$

Theorem: $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$

Corollary: $\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} = 0$

$= \lim_{x\to 0} \frac{\cos x - 1}{x}$

Example: Find the following limits

$\lim_{x\to 0}\cos(\sin x)$ $= \cos\left(\lim_{x\to 0}\sin x\right) = \cos(0) = 1$

$\lim_{x\to 0}\frac{\sin x}{3x}$ $= \lim_{x\to 0}\left[\left(\frac{\sin x}{x}\right)\left(\frac{1}{3}\right)\right] = \lim_{x\to 0}\left(\frac{\sin x}{x}\right)\cdot\lim_{x\to 0}\left(\frac{1}{3}\right) = (1)\left(\frac{1}{3}\right) = \frac{1}{3}$

$\lim_{x\to 0}\frac{\sin 7x}{2x}$ $= \lim_{x\to 0}\frac{\sin 7x}{7x}\cdot\frac{7}{2} = \lim_{x\to 0}\frac{\sin 7x}{7x}\cdot\lim_{x\to 0}\frac{7}{2} = 1\cdot\frac{7}{2} = \frac{7}{2} = 3.5$

$\lim_{x\to 0}\frac{\sin 4x}{\sin 8x}$ $= \lim_{x\to 0}\left(\frac{\sin 4x}{4x}\right)\cdot\left(\frac{8x}{\sin 8x}\right)\left(\frac{4}{8}\right) = 1\cdot 1\cdot\frac{1}{2} = \frac{1}{2}$

$\lim_{x\to 0}\frac{\cos x - 1}{\sin x}$ $= \lim_{x\to 0}\left(\frac{\cos x - 1}{x}\right)\left(\frac{x}{\sin x}\right) = 0\cdot 1 = 0$

$\lim_{x\to 0}\frac{\tan^2 4x}{x^2}$ $=$ 16 clicker question

$\lim_{x\to 0}\left(\frac{\tan 4x}{x}\right)^2 = \lim_{x\to 0}\left(\frac{\tan 4x}{4x}\cdot 4\right)^2 \Rightarrow \lim_{\theta\to 0}\frac{\tan\theta}{\theta}$?

$\lim_{\theta\to 0}\frac{\tan\theta}{\theta} = \lim_{\theta\to 0}\frac{\sin\theta}{\theta\cos\theta} = \lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\lim_{\theta\to 0}\frac{1}{\cos\theta} = 1\cdot 1 = 1$

KNOW
$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$ and
$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} = 0$

Exam 1 next Thursday (Feb 13)
Location TBA – 7:30 PM
Covers Ch 1, Ch 3 + 4.1