4.4 Chain Rule and Higher Derivatives

The Chain Rule:

If $g'(x)$ and $f'\left(g(x)\right)$ exist and $F(x) = f\left(g(x)\right)$, then

$$F'(x) = f'\left(g(x)\right) \cdot g'(x)$$

In Leibniz notations, if $y = f(u)$ and $u = g(x)$ are differentiable, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Example: Find $f'\left(x\right)$ when $f\left(x\right) = \frac{1}{\sqrt{7-3x}}$

Example: Find y' with and without the chain rule at $x = 1$.

$y = u - u^2, \quad u = \sqrt{x} + \sqrt[3]{x}$

Commonly use the chain rule and the power rule together,

$$\frac{d}{dx}\left(u^n\right) = nu^{n-1}\frac{du}{dx}$$

Example: Find the derivatives of the following functions

$$F(x) = \left(x^3 - 5x\right)^4$$

$$g(t) = \left(6t^2 + 5\right)^3\left(t^3 - 7\right)^4$$

$$y = \left(\frac{x}{2\left(x^2 - 1\right)^2 - 1}\right)^2$$

Example: Find the equation of the tangent line to the curve at the given point.

$y=\sqrt{x+(1/x)}, \quad (1,\sqrt{2})$

Example: Suppose that $w = u \circ v$ and $u(0) = 1$, $v(0) = 2$, $u'(0) = 3$, $u'(2) = 4$, $v'(0)=5$, and $v'(2) = 6$. Find $w'(0)$.

4.4.2 Implicit Differentiation

Example: Find $\frac{dy}{dx}$ when $\sqrt{x}+\sqrt{y}=4$

When $y = f(x)$, then y is an explicit function of x and $y' = dy/dx$ is straightforward. When this is not the case, we can use implicit differentiation which consists of differentiating both sides of the relation with respect to x and solving for y'.

$$\sqrt{x}+\sqrt{y}=4$$

Example: Find $\frac{dy}{dx}$ when

$$\sqrt{xy}-2x=\sqrt{y}$$

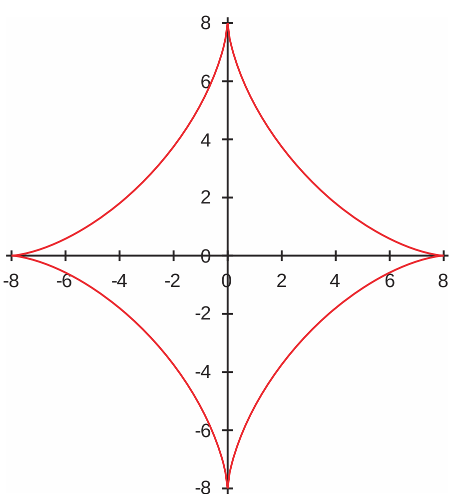
$$y^5+3x^2y^2+5x^4=12$$

Example: Regard y as the independent variable and x as the dependent variable and find dx/dy for

$$\left(x^2+y^2\right)^2=ax^2y$$

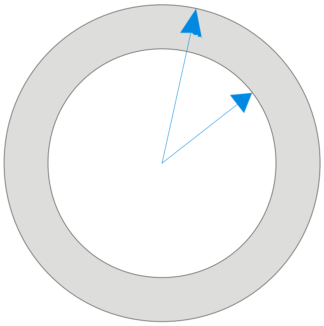
Example: If $\left[g(x)\right]^2+12x=x^2g(x)$ and $g(3)=4,$ find $g'(3)$.

Example: Find an equation of the tangent line to the curve at the given point.
$x^{2/3}+y^{2/3}=4$ at $\left(-3\sqrt{3},1\right)$



4.4.3 Related Rates

Example: Consider a circle of radius $r(t)$ that is increasing its radius at a rate of 5mm/sec. How is the area of the circle changing at time $t = 1$?



Example:
If $x^2 + 3xy + y^2 = 1$ and $dy/dt = 2$, find dx/dt when $y = 1$.

Example: If a spherical snowball melts so that its surface area decreases at a rate of 1 cm^2/min, find the rate at which the diameter decreases when the diameter is 10 cm.

Example: A spotlight on the ground shines on a wall 12 meters away. If a man 2 meters tall walks from the spotlight towards the building at a speed of 1.6 m/s, how fast is his shadow on the building decreasing when he is 4 meters from the building?

Example: At noon, ship A is 150 km west of ship B. Ship A is sailing east at 35 km/hr and ship B is sailing north at 25 km/hr. How fast is the distance between the ships changing at 4:00 PM?

Example: A storm is 50 miles offshore and its path is perpendicular to a straight shoreline. It is approaching the show at a rate of 4 mph. A van traveling along the shoreline wants to stay exactly 50 miles from the storm and remain along the shoreline. The van starts at the point on the shoreline in the path of the storm.

Find a formula for the speed that the van must maintain to remain 50 miles from the storm. What is the speed of the van when the storm is 40 miles from shore?

4.4.4 Higher Derivatives

The second derivative of the function $y = f(x)$ is

$$y'' = f''(x) = \frac{d}{dx}\left(f'(x)\right) = \frac{d}{dx}\left(\frac{d}{dx}f(x)\right) = \frac{d^2y}{dx^2} = D^2 f(x) = D_x^2 f(x)$$

The third derivative of the function $y = f(x)$ is

$$y''' = f'''(x) = \frac{d}{dx}\left(f''(x)\right) = \frac{d^3y}{dx^3} = D^3 f(x) = D_x^3 f(x)$$

The n^{th} derivative of the function $y = f(x)$ is

$$y^{(n)} = f^{(n)}(x) = \frac{d}{dx}\left(f^{(n-1)}(x)\right) = \frac{d^ny}{dx^n} = D^n f(x) = D_x^n f(x)$$

Example: Find the second derivative of the following functions

$$g(u) = \frac{1}{\sqrt{1-u}}$$

Example: Find a formula for $f^{(n)}(x)$ when $f(x) = \dfrac{1}{(1-x)^2}$

The instantaneous rate of change of the velocity is the **acceleration**. So if $s(t)$ is the position of an object at time t, the acceleration is
$a(t) = v'(t) = s''(t)$

Example: Given the position in meters at time t in seconds of an object is given by $s = 2t^3 - 9t^2$, find the times when the acceleration is zero. At the times when the acceleration is zero, where is the object and what is the object's velocity?

Example: Find $f''(x)$ if $f(x) = g\left(x^3\right) + \left(g(x)\right)^3$

Example: Find y'' by implicit differentation for $\sqrt{x} + \sqrt{y} = 1$.

Example: Find f' and f''. Sketch f, f' and f'' and determine their domains.

$f(x) = \left|x^2 - x\right|$