Chapter 5: Applications of Differentiation

5.1: Extrema and the Mean-Value Theorem

Absolute Extreme values of f

- A function *f* has an absolute (or global) maximum at *c* if $f(c) \geq f(x)$ for all *x* in the domain of *D*. The number $f(c)$ is called the maximum value of *f* on *D*.
- A function *f* has an absolute (or global) minimum at *d* if $f(d) \leq f(x)$ for all *x* in the domain of *D*. The number $f(d)$ is called the minimum value of *f* on *D*.

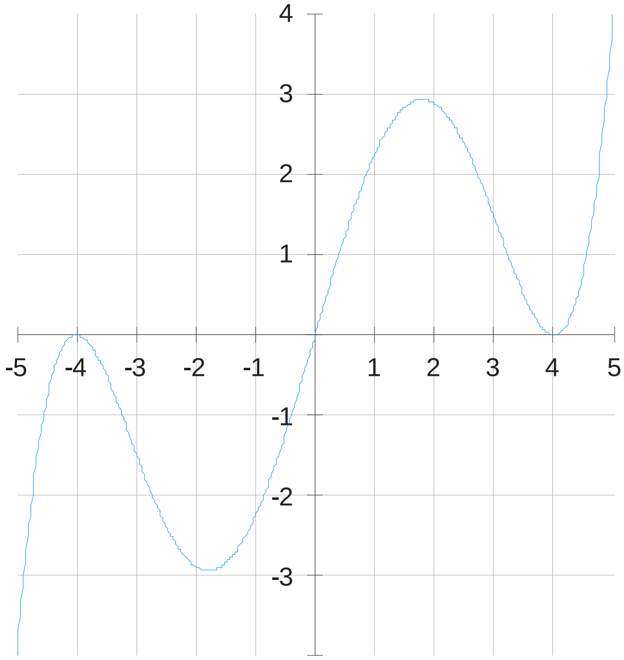
Local Extreme values of f

- A function *f* has a local (or relative) maximum at *c* if $f(c) \geq f(x)$ when *x* is near *c*.
- A function *f* has a local (or relative) minimum at *d* if $f(d) \leq f(x)$ when *x* is near *c*.

The Extreme Value Theorem

If *f* is continuous on a closed interval $[a,b]$, then *f* attains an absolute maximum value of $f(c)$ and an absolute minimum value $f(d)$ at some numbers *c* and *d* in $[a,b]$

Example: The graph of the function f with domain $[-5,5]$ is shown.



(a) What are the absolute extrema?

(b) What are the local extrema?

Example: Find all absolute and local extrema for the following functions

$f(x)=1-x^2,\ 0\le x\le 1$

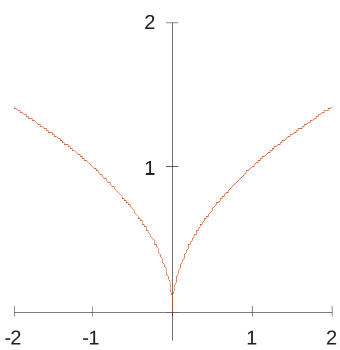
$f(x)=1-x^2,\ -1\le x\le \frac{1}{2}$

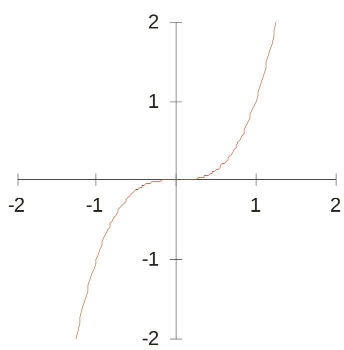
$$f(x)=\frac{1}{x},\ 0<x<1$$

$$f(x)=\begin{cases} x^2 & \text{for } -1\le x<0 \\ 2-x^2 & \text{for } 0\le x\le 1 \end{cases}$$

Fermat's Theorem

If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c)=0$.





Critical Numbers

A **critical number** (or value) of a function f is a number c in the domain of f such that either $f'(c)=0$ or $f'(c)$ does not exist.

Example: Find all critical numbers for the following functions

$$f(x) = x^3 + 6x^2 + 3x - 1$$

$$f(x) = \left|x^2 - 1\right|$$

$$f(x) = \frac{x+1}{x^2 + x + 1}$$

$$f(x) = x + \sin x$$

$$f(x) = xe^{2x}$$

$$f(x) = \sqrt[3]{x^2 - x}$$

Finding Absolute Extrema

To find the absolute extrema of a continuous function f on a closed interval $[a,b]$,

1. Find all critical values c_1, c_2, etc of f in the interval (a,b)
2. Find the values of $f(c_1)$, $f(c_2)$ etc along with $f(a)$ and $f(b)$
3. The absolute maxima is the largest of the values found in steps 1 and 2.
4. The absolute minima is the smallest of the values found in steps 1 and 2.

Example: Find the absolute extrema for the following functions

i. $f(x) = x^3 - 12x + 1, \quad [-3, 5]$

ii. $f(x) = -x^3 + 27x + 1, \quad [0, 4]$

iii. $f(x) = x - 2\cos x, \quad [0, \pi]$

iv. $f(x) = x - 2\cos x, \quad [-\pi, \pi]$

v. $f(x) = \dfrac{x}{x+1}, \quad [1, 2]$

vi. $f(x) = \dfrac{x}{x+1}, \quad [-2, 2]$

vii. $f(x) = \dfrac{\ln x}{x}, \quad [1, 3]$

<u>Mean Value Theorem</u>

If *f* is continuous on a closed interval $[a,b]$ and differentiable on the interval (a,b), then there exists a number *c*, where $a<c<b$ such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

Example: Given $f(x)=4-x^2$, show $f(x)$ satisfies the *MVT* on $[1,2]$ and find all values of *c* that satisfy the conclusion of the *MVT*.

Example: Given that $1\le f'(x)\le 4$ for all *x* in the interval $[2,5]$, prove that $3\le f(5)-f(2)\le 12$ given that *f* is continuous on [2,5] and differentiable on (2,5)