5.2: Monotonicity and Concavity

A function f defined on an interval I is called **(strictly) increasing** on I if

$$f\left(x_1\right) < f\left(x_2\right) \text{ whenever } x_1 < x_2 \text{ in } I$$

and is called **(strictly) decreasing** on I if

$$f\left(x_1\right) < f\left(x_2\right) \text{ whenever } x_1 < x_2 \text{ in } I$$

A function that is always increasing or always decreasing is called **monotonic**.

First Derivative Test for Monotonicity

Suppose f is continuous on $[a,b]$ and differentiable on (a,b)

- If $f'\left(x\right) > 0$ for all $x \in (a,b)$, then f is **increasing** on $[a,b]$
- If $f'\left(x\right) < 0$ for all $x \in (a,b)$, then f is **decreasing** on $[a,b]$

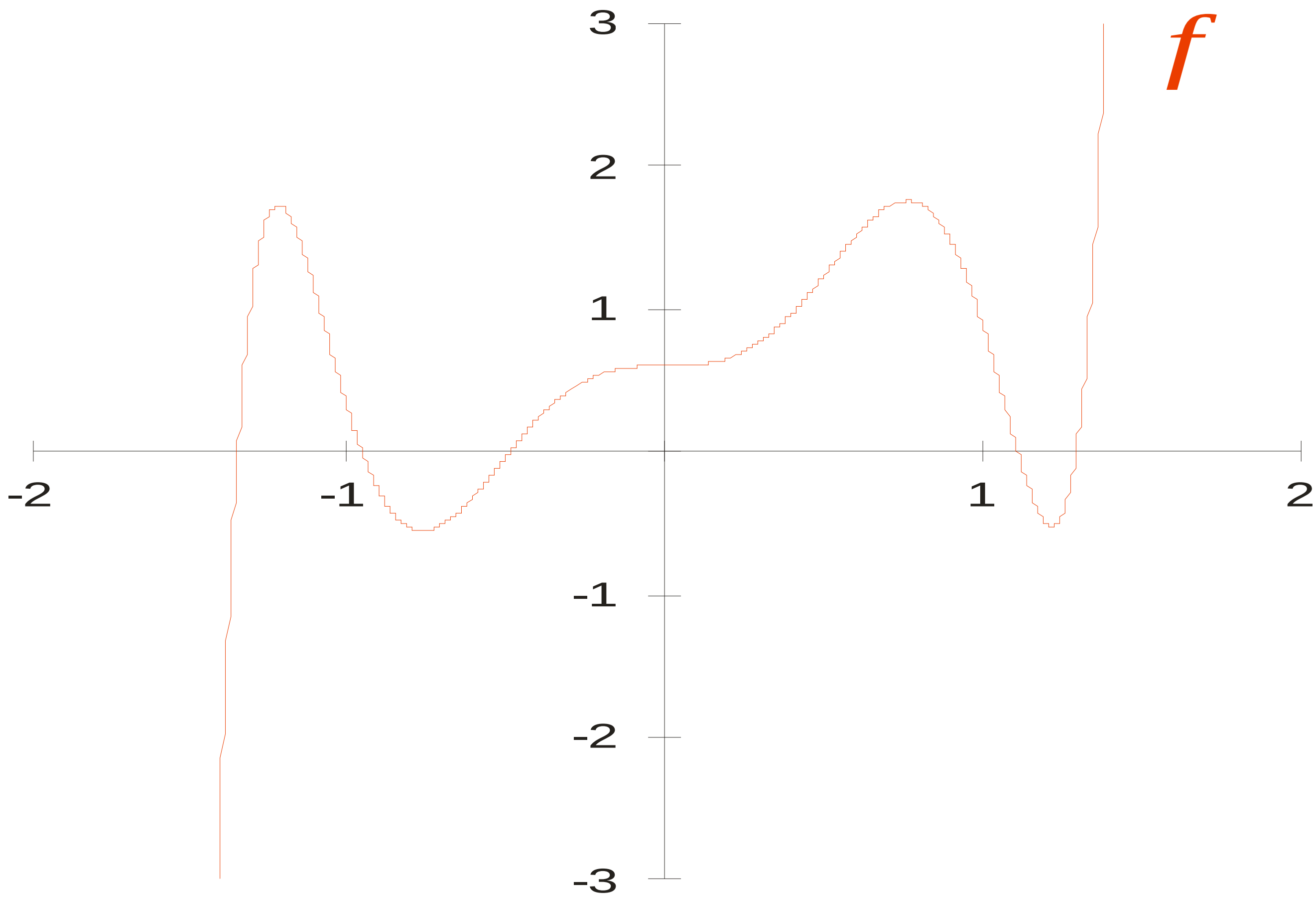
Second Derivative Test for Concavity

Suppose f is twice differentiable on an open interval I

- If $f''\left(x\right) > 0$ for all $x \in I$, then f is **concave up** on I
- If $f''\left(x\right) < 0$ for all $x \in I$, then f is **concave down** on I

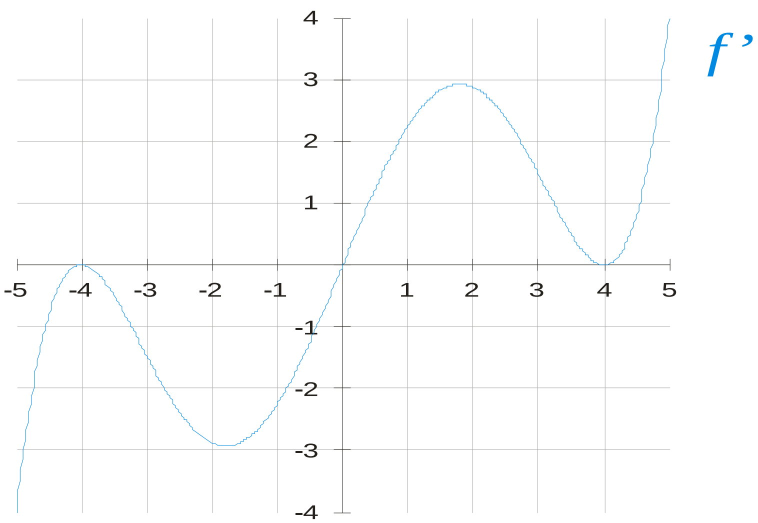
A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

An **inflection point** of a function f is the point where a function changes concavity.



Where is the function increasing? decreasing?
Where does the function have a local maximum? local minimum?
Where is the function concave up? concave down?
Where are the critical numbers and inflection points?

Example: The graph of the **derivative** of *f* is shown.



(a) Where is the function increasing or decreasing?

(b) Where might the function have a local maximum or minimum?

(c) Where is the function concave up or concave down?

(d) Where are the inflection points?

(e) If $f(0) = 0$, sketch a possible graph of *f*.

Example: Sketch a graph of *f* satisfying the following conditions:

$$f'(x) > 0 \text{ on } (-\infty, 1) \text{ and } f'(x) < 0 \text{ on } (1, \infty)$$
$$f''(x) > 0 \text{ on } (-\infty, -2) \text{ and } (2, \infty)$$
$$f''(x) < 0 \text{ on } (-2, 2)$$
$$\lim_{x \to -\infty} f(x) = -2 \text{ and } \lim_{x \to \infty} f(x) = 0$$

Example: Determine where each function is increasing, decreasing, concave up, and concave down.

(a) $y = (3x - 1)^{1/3}$

(b) $y = \dfrac{-2}{x^2 + 3}$