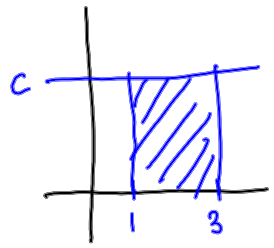
Clicker: How was recitation this week? (A) Great (B) Good (C) OK (D) Bad (E) Fail

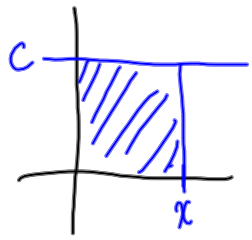
6.2: The Fundamental Theorem of Calculus

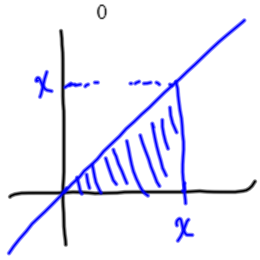
Example: Evaluate each integral by interpreting it in terms of area.

$f(t)=c$ $\quad$ $f(t)=c$ $\quad$ $f(t)=t$

(i) $\int_1^3 c\,dt = 2c$ $\qquad$ (ii) $\int_0^x c\,dt = cx$ $\qquad$ (iii) $\int_0^x t\,dt = \frac{1}{2}x^2$



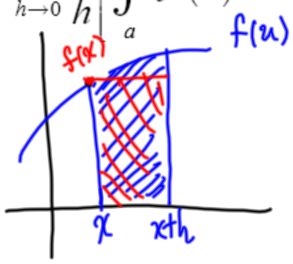




Let $F(x)=\int_a^x f(u)\,du$, how does the area change as x changes?

$$\frac{d}{dx}F(x)=\lim_{h\to 0}\frac{F(x+h)-F(x)}{h}=\lim_{h\to 0}\frac{1}{h}\left[\int_a^{x+h} f(u)\,du-\int_a^{x} f(u)\,du\right]$$

$$=\boxed{\lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(u)\,du}$$



What does $\int_x^{x+h} f(u)\,du$ mean graphically? It is about the area under the curve $f(u)$ from x to $x+h$. That is about $f(x)\cdot h$.

$$\frac{d}{dx}F(x)=\lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(u)\,du=\lim_{h\to 0}\frac{1}{h}\cdot f(x)\cdot h=\lim_{h\to 0} f(x)=f(x)$$

THE FUNDAMENTAL THEOREM OF CALCULUS, PART 1

If f is continuous on $[a,b]$, then the function F defined by

$$F(x) = \int_a^x f(u)\,du, \qquad a \le x \le b$$

is continuous on $[a,b]$, differentiable on (a,b) and $F'(x) = f(x)$. So $F(x)$ is an antiderivative of $f(x)$.

Example: Find the derivative of the given functions

(i) $g(x) = \int_{-1}^{x} \sqrt{t^3+1}\,dt$ (ii) $y = \int_{-1}^{x} \frac{2}{t^2+1}\,dt$

(i) $\sqrt{t^3+1}$ is cont so FTC says $\frac{d}{dx} g(x) = \frac{d}{dx}\int_{-1}^{x} \sqrt{t^3+1}\,dt = \sqrt{x^3+1}$

(ii) $\frac{2}{t^2+1}$ is cont, so FTC says $\frac{dy}{dx} = \frac{d}{dx}\int_{-1}^{x} \frac{2}{t^2+1} = \frac{2}{x^2+1}$

$(F(b)+c) - (F(a)+c) = F(b) - F(a)$

THE FUNDAMENTAL THEOREM OF CALCULUS, PART 2

If f is continuous on $[a,b]$, then $\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_a^b$

where F is any antiderivative of f.

Example: Evaluate the following definite integrals

(i) $\int_0^2 (w^3 - 1)^2\, dw$ (ii) $\int_{-2}^0 |x^2 - 1|\, dx$ (iii) $\int_0^{\pi/2} (\cos\theta + 2\sin\theta)\, d\theta$

$$(i) = \int_0^2 (w^6 - 2w^3 + 1)\, dw = \frac{w^7}{7} - 2\frac{w^4}{4} + w \Big|_0^2$$

$$= \left(\frac{2^7}{7} - \frac{1}{2}(2)^4 + 2\right) - \left(\frac{0^7}{7} - \frac{1}{2}(0)^4 + 0\right) = \frac{87}{7}$$

$$(ii) \int_{-2}^0 |x^2-1|\, dx = \int_{-2}^{-1} (x^2-1)\, dx + \int_{-1}^0 (1-x^2)\, dx = \left(\frac{x^3}{3} - x\right|_{-2}^{-1} + \left(x - \frac{x^3}{3}\right|_{-1}^0$$

$$= \left(\frac{(-1)^3}{3} - (-1)\right) - \left(\frac{(-2)^3}{3} - (-2)\right) + \left(0 - \frac{0^3}{3}\right) - \left(-1 - \frac{(-1)^3}{3}\right) = 2$$

$$(iii) \int_0^{\pi/2} (\cos\theta + 2\sin\theta)\, d\theta = \left(\sin\theta - 2\cos\theta\right|_0^{\pi/2} = \left(\sin(\pi/2) - 2\cos(\pi/2)\right) - \left(\sin(0) - 2\cos(0)\right)$$

$$= \underline{\quad 3 \quad}$$
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If the general antiderivative of $f(x)$ is $F(x)+C$, this can be written as

$$F(x)+C = C + \int_a^x f(u)\,du = \int f(x)\,dx$$

Where $\int f(x)\,dx$ is called an **indefinite integral**.

Example: Compute the following indefinite integrals

(i) $\int \left(x^{3/5}+x^{5/3}\right)dx$ (ii) $\int \cos\left(\frac{2-4x}{5}\right)dx$ (iii) $\int 2e^{-x/4}\,dx$

(iv) $\int 4^{-x}dx$ (v) $\int \frac{5}{\sqrt{1-x^2}}dx$

(i) $\frac{5}{8}x^{8/5} + \frac{3}{8}x^{8/3} + C$

(ii) $u = \frac{2-4x}{5}$ imp. diff $\Rightarrow du = -\frac{4}{5}dx \Rightarrow dx = -\frac{5}{4}du$

$\int \cos\left(\frac{2-4x}{5}\right)dx = \int \cos(u)\left(\frac{-5}{4}\right)du = \frac{-5}{4}\int \cos u\,du = \left(\frac{-5}{4}\right)\sin u + C$

$= \left(\frac{-5}{4}\right)\sin\left(\frac{2-4x}{5}\right) + C$

(iii) $\int 2e^{-x/4}dx \Rightarrow u = -x/4$, $du = -dx/4 \Rightarrow dx = -4\,du$

$\int 2e^{u}(-4)\,du = -8\int e^u du = -8e^u + C = -8e^{-x/4} + C$

(iv) $\int 4^{-x}dx \Rightarrow u = -x \Rightarrow du = -dx \Rightarrow dx = -du$

$\int 4^u(-1)\,du = -\int 4^u du = -\frac{4^u}{\ln 4} + C = -\frac{4^{-x}}{\ln 4} + C$

(v) $5\int \frac{1}{\sqrt{1-x^2}}dx = 5\arcsin x + C$