Clicker: Have you done the course evaluation? (A) Yes (B) No, but I will later (C) No, and I won't bother

7.1: The Substitution Rule

Let $f(x)=e^{x^2}$, then $f'(x)=2xe^{x^2}$. What does that mean for $\int xe^{x^2}\,dx$?

$$\int xe^{x^2}dx = \frac{1}{2}\int 2xe^{x^2}dx = \frac{1}{2}e^{x^2}+C$$

If you have an integrand of the form $f'\left[g(x)\right]\cdot g'(x)$, then

$$\int f'\left[g(x)\right]g'(x)\,dx = f\left[g(x)\right]+C$$

General Indefinite Integral Formulas

$$\int\left[f(x)\right]^n f'(x)\,dx = \frac{\left[f(x)\right]^{n+1}}{n+1}+C,\quad n\neq -1$$

$$\int e^{f(x)}f'(x)\,dx = e^{f(x)}+C$$

$$\int \frac{1}{f(x)}f'(x)\,dx = \ln\left|f(x)\right|+C$$

$$\int f'(x)\sin\left[f(x)\right]dx = -\cos\left[f(x)\right]+C$$

$$\int f'(x)\cos\left[f(x)\right]dx = \sin\left[f(x)\right]+C$$

$$\int f'(x)\sec^2\left[f(x)\right]dx = \tan\left[f(x)\right]+C$$

Example: Evaluate $\int\underbrace{\left(x^2+10x+7\right)}_{f(x)}{}^4\underbrace{(2x+10)}_{f'(x)=2x+10}dx$

$$\int f(x)^4\cdot f'(x)\,dx = \frac{f(x)^5}{5}+C = \frac{\left(x^2+10x+7\right)^5}{5}+C$$

Integration by Substitution Rule for Indefinite Integrals:

$$\int f\left[g(x)\right]g'(x)\,dx = \int f(u)\,du \text{ where } u = g(x)$$

Example: Evaluate the following integrals

(i) $\int 3x^9\sqrt{4x^{10}-7}\,dx$ (ii) $\int \frac{\ln t}{t}dt$ (iii) $\int \tan x\,dx$ (iv) $\int \frac{x}{\sqrt{x-9}}dx$

(i) $\int 3x^9\sqrt{4x^{10}-7}\,dx$: $u = 4x^{10}-7$; $du = 40x^9 dx$ or $\frac{du}{dx} = 40x^9$

$\hookrightarrow dx = \frac{du}{40x^9}$

$= \int 3\cancel{x^9}\sqrt{u}\,\frac{du}{40\cancel{x^9}} = \frac{3}{40}\int u^{1/2}du$

$= \frac{3}{40}\left(\frac{2}{3}u^{3/2}\right)+C = \frac{1}{20}(u)^{3/2}+C = \frac{1}{20}\left(4x^{10}-7\right)^{3/2}+C$

(ii) $\int \frac{\ln t}{t}dt$, $u = \ln t$, $du = \frac{1}{t}dt$

$= \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}(\ln t)^2 + C$

(iii) $\int \frac{\sin x}{\cos x}dx \rightarrow u = \cos x \rightarrow du = -\overbrace{\sin x\,dx} \rightarrow \sin x\,dx = -du$

$= \int \frac{-du}{u} = -\ln|u| + C = -\ln|\cos x| + C$

(iv) $\int \frac{x\,dx}{\sqrt{x-9}} \rightarrow u = x-9$, $x = 9+u$, $du = dx$

$\int \frac{9+u}{\sqrt{u}}du = \int 9u^{-1/2} + u^{1/2}\,du = 9\cdot 2\cdot u^{1/2} + \frac{2}{3}u^{3/2} + C = 18(x-9)^{1/2} + \frac{2}{3}(x-9)^{3/2} + C$

Integration by Substitution Rule for Definite Integrals:

$$\int_a^b f\left[g(x)\right]g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du \text{ where } u = g(x)$$

Example: Evaluate the following integrals

(i) $\int_{-1}^{4} x\sqrt{x+5}\,dx$ (ii) $\int_0^{\pi/3} \frac{\sin x}{\cos^2 x}dx$ (iii) $\int_{\ln 4}^{\ln 7} \frac{e^x}{\left(e^x-3\right)^2}dx$ (iv) $\int_1^2 x^5\sqrt{x^3+2}\,dx$

(i) $\int_{x=-1}^{x=4} x\sqrt{x+5}\,dx$: $u=x+5$, $x=u-5$, $du=dx$, If $x=-1$, $u=4$; If $x=4$, $u=9$

$= \int_4^9 (u-5)\sqrt{u}\,du = \int_4^9 u^{3/2}-5u^{1/2}\,du = \left(\frac{2}{5}u^{5/2} - 5\cdot\frac{2}{3}u^{3/2}\right)\Big|_{u=4}^{u=9}$

$\int_{x=-1}^{x=4}\left(u^{3/2}-5u^{1/2}\right)du$

$= \left(\frac{2}{5}9^{5/2} - \frac{10}{3}\cdot 9^{3/2} - \frac{2}{5}\cdot 4^{5/2} + \frac{10}{3}\cdot 4^{3/2}\right) = \underline{\qquad}$ clicker (1 decimal place)

$\int_0^{\pi/3} \frac{\sin x}{\cos^2 x}dx$, $u=\cos x \rightarrow du=-\sin x\,dx$, if $x=0$, $u=1$
if $x=\pi/3$, $u=1/2$

$\int_1^{1/2} \frac{-du}{u^2} = +\;u^{-1}\Big|_1^{1/2} = \underline{\qquad}$ clicker

$\frac{d}{dx}x^{-1} = -\frac{1}{x^2} = -x^{-2}$

$\int_{\ln 4}^{\ln 7} \frac{e^x}{(e^x-3)^2}dx$; $u=e^x-3$; $du=e^x\,dx$

If $x=\ln 4$, $u=e^{\ln 4}-3=1$
$x=\ln 7$, $u=e^{\ln 7}-3=4$

$= \int_1^4 \frac{du}{u^2} = \left(-\frac{1}{u}\right|_1^4 = -\frac{1}{4}-\left(-\frac{1}{1}\right) = -\frac{1}{4}+1 = .75$

$\int_1^2 x^5\sqrt{x^3+2}\,dx$: $u=x^3+2$, $du=3x^2dx$, $x^3=u-2$
If $x=1$, $u=3$; If $x=2$, $u=10$

$\int_{u=3}^{10} x^3\sqrt{u}\;\;x^2dx = \int_3^{10}\sqrt{u}\,(u-2)\frac{du}{3} = \frac{1}{3}\int_3^{10}\left(u^{3/2}-2u^{1/2}\right)du = \underline{28.34}$ clicker 2 decimal places

$= \frac{1}{3}\left(\frac{2}{5}\cdot u^{5/2} - 2\cdot\frac{2}{3}\cdot u^{3/2}\right|_3^{10} = \frac{1}{3}\left(\frac{2}{5}\cdot 10^{5/2} - \frac{4}{3}\cdot 10^{3/2} - \frac{2}{5}\cdot 3^{5/2} + \frac{4}{3}\cdot 3^{3/2}\right)$