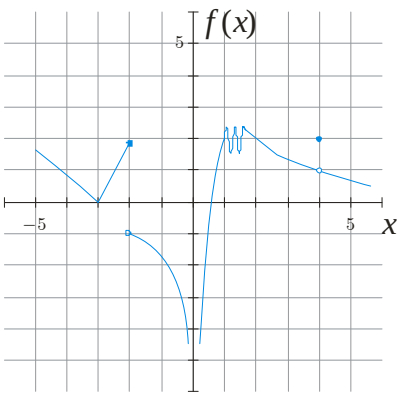
The Limit of a Function (Section 2.2)

The limit of $f(x)$ as x approaches a is L if we can make the values of $f(x)$ arbitrarily close to L as x is close to a but not equal to a.

$$\lim_{x \to a} f(x) = L$$



What can we say about the $f(x)$ in the graph on the left?

$\lim_{x \to a^-} f(x)$ is the left-hand limit

$\lim_{x \to a^+} f(x)$ is the right-hand limit

$$\boxed{\lim_{x \to a} f(x) = L \text{ if and only if } \lim_{x \to a^-} f(x) = L \text{ and } \lim_{x \to a^+} f(x) = L}$$

Let f be a function defined on both sides of a, except possibly at a itself. Then $\lim_{x \to a} f(x) = \infty$ means that the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a.

Similarly, $\lim_{x \to a} f(x) = -\infty$ when the values of $f(x)$ can be made arbitrarily negatively large by taking x sufficiently close to a.

The line $x = a$ is called a vertical asymptote of the function $f(x)$ if at least one of the following is true:

$$\lim_{x \to a^-} f(x) = \infty \text{ or } -\infty \qquad \lim_{x \to a^+} f(x) = \infty \text{ or } -\infty$$

EXAMPLE 1

Using the graph of $f(x)$ find the following limits, if they exit

a) $\lim_{x \to -5} f(x)$

b) $\lim_{x \to -3} f(x)$

c) $\lim_{x \to -2} f(x)$

d) $\lim_{x \to 0} f(x)$

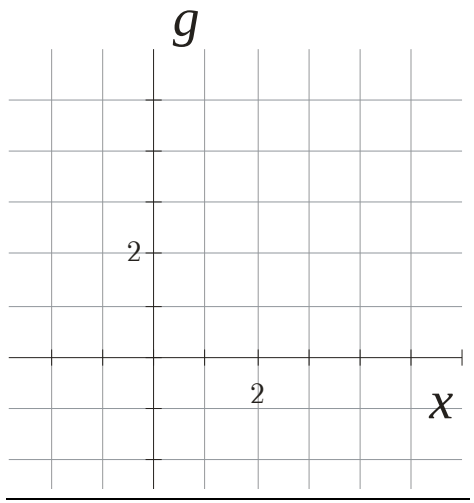
e) $\lim_{x \to 1} f(x)$

f) $\lim_{x \to 4} f(x)$

EXAMPLE 2

Sketch the graph of the function

$$g(x)=\begin{cases} 2-x & \text{if } x<-1 \\ x & \text{if } -1\le x<1 \\ 4 & \text{if } x=1 \\ 4-x & \text{if } x>1 \end{cases}$$



Use the graph to find the following limits, if they exist:

a) $\lim\limits_{x\to -1} g(x)$

b) $\lim\limits_{x\to 1} g(x)$

EXAMPLE 3

Evaluate the function $g(x)$ at the given values and then guess the value of $\lim\limits_{x\to 2^+} g(x)$.

$$g(x)=\frac{1-x^2}{x^2+3x-10}$$

$g(3)=$ $\qquad$ $g(2.001)=$

$g(2.1)=$ $\qquad$ $g(2.0001)=$

$g(2.01)=$ $\qquad$ $g(2.00001)=$

EXAMPLE 4

Evaluate the function $g(x)$ at the given values and then guess the value of $\lim\limits_{x\to 0^-} g(x)$.

$$g(x)=\frac{\cos x-1}{\sin x}$$

$g(-1)=$ $\qquad$ $g(-0.1)=$

$g(-0.5)=$ $\qquad$ $g(-0.01)=$

$g(-0.3)=$ $\qquad$ $g(-0.001)=$

EXAMPLE 5

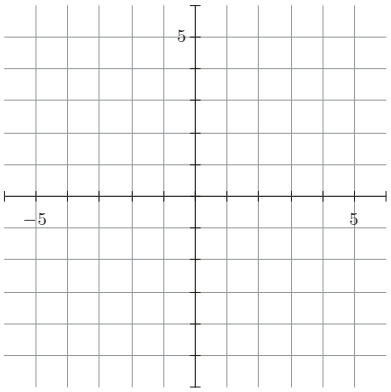
Find the infinite limits

a) $\lim\limits_{x\to 0}\dfrac{x-1}{x^2(x+2)}$

b) $\lim\limits_{x\to 5}\dfrac{6}{5-x}$

EXAMPLE 6

Find the vertical asymptotes of $y=\dfrac{x}{x^2-x-2}$ and sketch the graph.



Limits of Vector Functions: $\lim\limits_{t\to a}\boldsymbol{r}(t)=\boldsymbol{b}$

If $\boldsymbol{r}(t)=\langle f(t),g(t)\rangle$, then $\lim\limits_{t\to a}\boldsymbol{r}(t)=\left\langle \lim\limits_{t\to a} f(t),\lim\limits_{t\to a} g(t)\right\rangle$

EXAMPLE 7

Given $\boldsymbol{r}(t)=\left\langle \dfrac{t^2-4t+4}{t-2},\dfrac{t^2-4}{2t-4}\right\rangle$, find $\lim\limits_{t\to 2}\boldsymbol{r}(t)$ by finding

$\boldsymbol{r}(1.2)=$ $\boldsymbol{r}(2.8)=$

$\boldsymbol{r}(1.5)=$ $\boldsymbol{r}(2.5)=$

$\boldsymbol{r}(1.8)=$ $\boldsymbol{r}(2.2)=$

$\boldsymbol{r}(1.9)=$ $\boldsymbol{r}(2.1)=$

$\boldsymbol{r}(1.99)=$ $\boldsymbol{r}(2.01)=$