Implicit Differentiation (Section 3.6)

EXAMPLE 1

Find $\frac{dy}{dx}$ when

$\sqrt{y} = 4 - \sqrt{x}$

$$\sqrt{x} + \sqrt{y} = 4$$

$(\sqrt{y})^2 = (4-\sqrt{x})^2 \Rightarrow y = (4-\sqrt{x})^2$

$y' = 2(4-\sqrt{x})(-\frac{1}{2}x^{-1/2})$

$\frac{d}{dx}(x^{1/2} + y^{1/2}) = \frac{d}{dx}(4) \Rightarrow \frac{1}{2}x^{-1/2}\frac{dx}{dx} + \frac{1}{2}y^{-1/2}\frac{dy}{dx} = 0$

$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}y' = 0 \quad \frac{1}{2\sqrt{y}}y' = -\frac{1}{2\sqrt{x}} \Rightarrow y' = \frac{-1}{2\sqrt{x}} \cdot 2\sqrt{y}$

$= y' = \frac{-\sqrt{y}}{\sqrt{x}} = \frac{-4-\sqrt{x}}{\sqrt{x}}$

$= \frac{-4}{\sqrt{x}} + 1$

$$\sqrt{xy} - 2x = \sqrt{y}$$

$\frac{d}{dx}(x^{1/2}y^{1/2} - 2x)$

Pop Quiz

$$y^5 + 3x^2y^2 + 5x^4 = 12$$

find y'

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$$x\cos y + y\cos x = 1$$

$$\frac{d}{dx}(x\cos y + y\cos x) = \frac{d}{dx}(1) \rightarrow x\frac{d}{dx}\cos y + \cos y\frac{dx}{dx} + y\frac{d}{dx}\cos x + \cos x\frac{dy}{dx} = 0$$

$$x(-\sin y)\frac{dy}{dx} + \cos y\,(1) + y(-\sin x)\frac{dx}{dx} + \cos x\, y' = 0$$

$$-x\sin y\, y' + \cos y - y\sin x + y'\cos x = 0$$

$$y' = \frac{y\sin x - \cos y}{\cos x - x\sin y}$$

When $y = f(x)$, then y is an explicit function of x and $y' = dy/dx$ is straightforward. When this is not the case, we can use implicit differentiation which consists of differentiating both sides of the relation with respect to x and solving for y'.

EXAMPLE 2

Regard y as the independent variable and x as the dependent variable and find dx/dy for

x'

$$\left(x^2+y^2\right)^2 = ax^2y$$

$$\frac{d}{dy}(x^2+y^2)^2 = \frac{d}{dy}ax^2y \Rightarrow 2(x^2+y^2)\frac{d}{dy}(x^2+y^2) = ax^2\frac{d}{dy}y + y\frac{d}{dy}(ax^2)$$

$$2(x^2+y^2)(2x\,x' + 2y) = ax^2 + y\,2ax\,x'$$

$$\Rightarrow x' = \frac{ax^2 - 4y(x^2+y^2)}{4x(x^2+y^2) - 2axy}$$

EXAMPLE 3

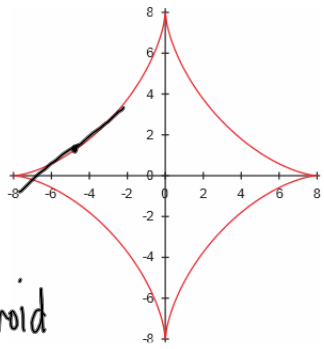
Find an equation of the tangent line to the curve at the given point.

$$x^{2/3}+y^{2/3} = 4 \text{ at } \left(\overset{\approx -5}{-3\sqrt{3}},1\right)$$

$$\frac{d}{dx}\left(x^{2/3}+y^{2/3}\right) = \frac{d}{dx}(4)$$

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0$$

$$y' = -\frac{\sqrt[3]{y}}{\sqrt[3]{x}} \quad \text{at } x=-3\sqrt{3},\ y=1$$



astroid

$$y' = -\frac{\sqrt[3]{1}}{\sqrt[3]{-3\sqrt{3}}} = -\frac{1}{-\left(3^{3/2}\right)^{1/3}} = +\frac{1}{\sqrt{3}} = m$$

$$-3 = -1\times 3 = -1\times(\sqrt{3})^2$$

$$y - y_1 = m(x - x_1)$$

$$y - 1 = \frac{1}{\sqrt{3}}\left(x - (-3\sqrt{3})\right)$$

$$y = \frac{x}{\sqrt{3}} + 3 + 1 = \frac{x}{\sqrt{3}} + 4$$

EXAMPLE 4

$x=3$

If $[g(x)]^2 + 12x = x^2 g(x)$ and $g(3) = 4$, find $g'(3)$.

$2g(x)g'(x) + 12 = x^2 g'(x) + g(x)\cdot 2x$

$\underbrace{2(4)}_{8} g'(3) + 12 = (3)^2 g'(3) + \underbrace{4 * 2 * 3}$

$g'(3) = -12$

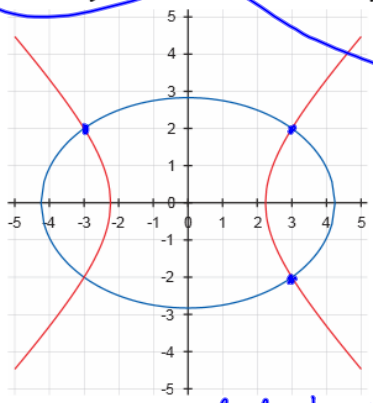
$m_1 \times m_2 = -1$

Two curves are called orthogonal if at each point of intersection their tangent lines are perpendicular. Two families of curves are orthogonal trajectories of each other if every curve in one family is orthogonal to every curve in the other family.

EXAMPLE 5

Show that the given curves are orthogonal

$x^2 - y^2 = 5, \quad 4x^2 + 9y^2 = 72$



① find the intersection points

$y^2 = x^2 - 5$

$4x^2 + 9(x^2-5) = 72 \rightarrow 13x^2 = 117 \rightarrow x^2 = 9$

$x = \pm 3$

$y^2 = x^2 - 5 = 9 - 5 = 4 \Rightarrow y = \pm 2$

② find slope of the tangent lines

#1 $2x - 2yy' = 0 \rightarrow y_1' = x/y$

#2 $8x + 18yy' = 0 \rightarrow y_2' = \frac{-4x}{9y}$

$(3,2) \Rightarrow m_1 = 3/2, \quad m_2 = -\frac{4(3)}{9(2)} = \frac{-2}{3} \Rightarrow \perp$ b/c $m_1 \times m_2 = -1$

$(-3,2)$

$(-3,-2)$

$(-3,2)$

all ok

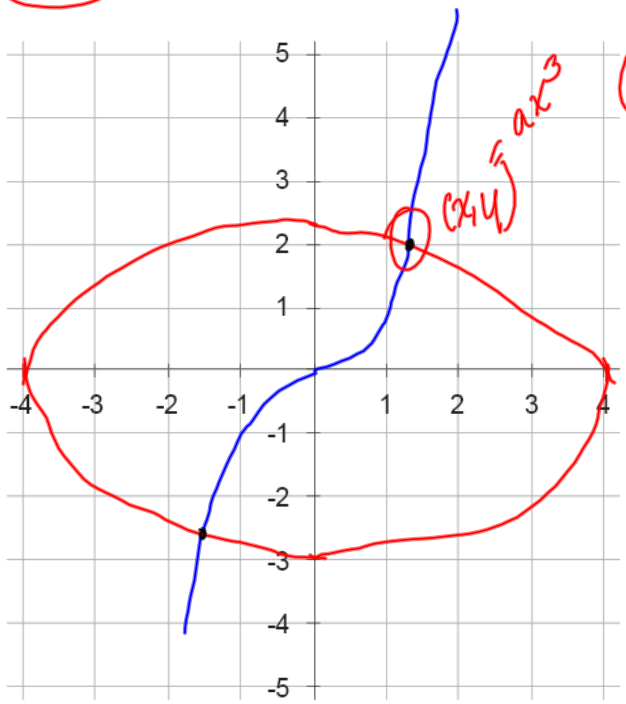
EXAMPLE 6

Show that the given families of curves are orthogonal trajectories of each other. Sketch.

$y = ax^3, \quad x^2 + 3y^2 = b$

$a = 1$
$b = 16$

note → intersect when same (x, y)



$y' = 3ax^2$

$\perp$?

$2x + 6y\,y' = 0$

$6y\,y' = -2x$

$y' = \frac{-2x}{6y} = \frac{-x}{3y}$

$y' = \frac{-x}{3(ax^3)} = \frac{-1}{3ax^2}$