Implicit Differentiation (Section 3.6)

EXAMPLE 1

Find $\frac{dy}{dx}$ when

$\sqrt{x}+\sqrt{y}=4 \rightarrow \sqrt{y}=4-\sqrt{x} \rightarrow y=(4-\sqrt{x})^2$

$y'=2(4-\sqrt{x})\frac{d}{dx}(4-x^{1/2})=2(4-\sqrt{x})(-\frac{1}{2}x^{-1/2})$

$=\frac{\sqrt{x}-4}{\sqrt{x}}=1-\frac{4}{\sqrt{x}}$

When $y=f(x)$, then y is an explicit function of x and $y'=dy/dx$ is straightforward. When this is not the case, we can use implicit differentiation which consists of differentiating both sides of the relation with respect to x and solving for y'.

$\sqrt{x}+\sqrt{y}=4 \rightarrow \sqrt{y}=4-\sqrt{x}$

$\frac{d}{dx}(x^{1/2}+y^{1/2})=\frac{d}{dx}(4)$

$\frac{1}{2}x^{-1/2}\frac{d}{dx}x+\frac{1}{2}y^{-1/2}\frac{d}{dx}y=0$

$\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}y'=0$

$\frac{1}{2\sqrt{y}}y'=-\frac{1}{2\sqrt{x}}$

$y'=-\frac{\sqrt{y}}{\sqrt{x}}=\frac{-(4-\sqrt{x})}{\sqrt{x}}=1-4/\sqrt{x}$

$\Rightarrow$ SAME RESULT!!

$\sqrt{xy}-2x=\sqrt{y}$ $\rightarrow$ much harder to solve for y. just diff $\rightarrow$

$\frac{d}{dx}(x^{1/2}y^{1/2})-2\frac{d}{dx}x=\frac{d}{dx}y^{1/2}$

$x^{1/2}\cdot\frac{1}{2}y^{-1/2}y'+y^{1/2}\cdot\frac{1}{2}x^{-1/2}-2=\frac{1}{2}y^{-1/2}y'$

$y'\left(\frac{\sqrt{x}}{2\sqrt{y}}\right)+\frac{\sqrt{y}}{2\sqrt{x}}-2=y'\left(\frac{1}{2\sqrt{y}}\right)$

$y'\left(\frac{\sqrt{x}-1}{2\sqrt{y}}\right)=2-\frac{\sqrt{y}}{2\sqrt{x}}=\frac{4\sqrt{x}-\sqrt{y}}{2\sqrt{x}}$

$y'=\left(\frac{4\sqrt{x}-\sqrt{y}}{2\sqrt{x}}\right)\left(\frac{2\sqrt{y}}{\sqrt{x}-1}\right)=\frac{4\sqrt{xy}-y}{x-\sqrt{x}}$

$y^5+3x^2y^2+5x^4=12$

$5y^4y'+3x^2\cdot 2yy'+y^2\,6x+20x^3=0$

$y'(5y^4+6x^2y)=-20x^3-6xy^2$

$y'=\frac{-20x^3-6xy^2}{(5y^4+6x^2y)}$

$x\cos y+y\cos x=1$

$\frac{d}{dx}(x\cos y)+\frac{d}{dx}(y\cos x)=\frac{d}{dx}1$

$x\frac{d}{dx}\cos y+\cos y\frac{d}{dx}x+y\frac{d}{dx}\cos x+\cos x\frac{d}{dx}y=0$

$x(-\sin y)y'+\cos y+y(-\sin x)+\cos x\,y'=0$

$y'(\cos x-x\sin y)=y\sin x-\cos y$

$\Rightarrow y'=\frac{y\sin x-\cos y}{\cos x-x\sin y}$

EXAMPLE 2

Regard y as the independent variable and x as the dependent variable and find dx/dy for

$$\left(x^2+y^2\right)^2 = ax^2y \Rightarrow \frac{d}{dy}\left(x^2+y^2\right)^2 = \frac{d}{dy}\left(ax^2y\right)$$

$$2\left(x^2+y^2\right)\frac{d}{dy}\left(x^2+y^2\right) = ax^2\frac{d}{dy}y + y\frac{d}{dy}ax^2$$

$$2\left(x^2+y^2\right)\left(2x\frac{dx}{dy}+2y\frac{dy}{dy}\right) = ax^2 + ay\,2x\frac{dx}{dy}$$

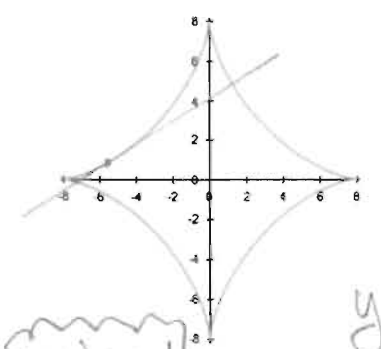
$$2\left(x^2+y^2\right)\left(2xx'+2y\right) = ax^2+2axyx'$$

$$\Rightarrow x' = \frac{ax^2-4y\left(x^2+y^2\right)}{4x\left(x^2+y^2\right)-2axy}$$

EXAMPLE 3

Find an equation of the tangent line to the curve at the given point.

$x^{2/3}+y^{2/3}=4$ at $\left(-3\sqrt{3},1\right) \approx (-5,1)$. Diff both sides



astroid

$$\frac{2}{3}x^{-1/3}+\frac{2}{3}y^{-1/3}y' = 0$$

$$\Rightarrow y' = -\sqrt[3]{y}/\sqrt[3]{x}$$

at $x=-3\sqrt{3}$, $y=1$

$$y' = -\sqrt[3]{1}/\sqrt[3]{-3\sqrt{3}} = \frac{-1}{\sqrt[3]{3^{3/2}}}$$

$$= \frac{1}{\sqrt{3}}$$

$$y-1 = \frac{1}{\sqrt{3}}\left(x+3\sqrt{3}\right)$$

$$y = \frac{1}{\sqrt{3}}x+4$$

EXAMPLE 4

If $\left[g(x)\right]^2+12x = x^2g(x)$ and $g(3)=4$, find $g'(3)$.

$$2g(x)g'(x)+12 = x^2g'(x)+g(x)\cdot 2x$$

$$x=3 \Rightarrow 2\cdot g(3)g'(3)+12 = 3^2g'(3)+g(3)\cdot 2(3)$$

$$8g'(3)+12 = 9g'(3)+24$$

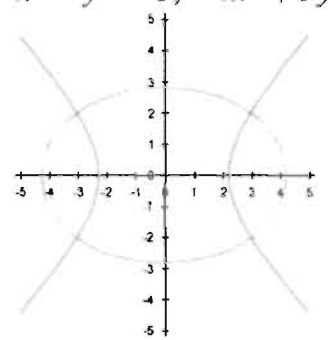
$$g'(3) = -12$$

Two curves are called orthogonal if at each point of intersection their tangent lines are perpendicular. Two families of curves are orthogonal trajectories of each other if every curve in one family is orthogonal to every curve in the other family.

EXAMPLE 5

Show that the given curves are orthogonal

$x^2-y^2=5, \quad 4x^2+9y^2=72$



① find the intersections

$y^2 = x^2-5$ plug in →

$4x^2+9\left(x^2-5\right) = 72$

$13x^2-45 = 72 \rightarrow x^2 = 9$ (+45, /13)

$\Rightarrow x = \pm 3$

$y^2 = x^2-5 = 9-5 = 4 \Rightarrow y = \pm 2$ 4 points

$$\frac{d}{dx}\left(x^2-y^2\right) = \frac{d}{dx}(5) \rightarrow 2x-2yy' = 0 \rightarrow y_1' = x/y$$

$$\frac{d}{dx}\left(4x^2+9y^2\right) = \frac{d}{dx}(72) \rightarrow 8x-18yy' = 0 \rightarrow y_2' = -\frac{4x}{9y}$$

② $(3,2)$ $y_1' = 3/2$, $y_2' = -\frac{4(3)}{9(2)} = -\frac{2}{3} \Rightarrow \perp$

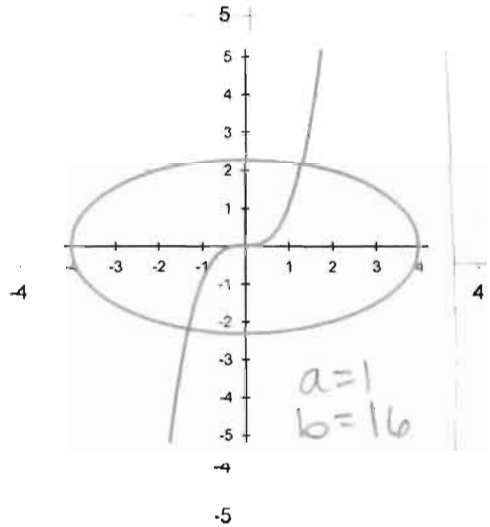
(etc)

EXAMPLE 6

Show that the given families of curves are orthogonal trajectories of each other. Sketch.

$y = ax^3, \quad x^2 + 3y^2 = b$

To sketch,
let $a=1, b=16$
$y_1 = x^3$
$x^2 + 3y_2^2 = 16$



$y_1' = 3ax^2$

$2x + 6yy_2' = 0$

$6yy_2' = -2x$

$y_2' = -\frac{x}{3y}$

at intersection, same (x, y)

$\Rightarrow y_2' = -\frac{x}{3(\underbrace{ax^3}_{=y})} = -\frac{1}{3ax^2}$

are $\perp$
since $m_1 \times m_2 = -1$