The Fundamental Theorem of Calculus (Section 6.4)

At the end of chapter 5 we introduced the concept of the antiderivative:

Antiderivatives
The function $F(x)$ an antiderivative of $f(x)$ if $F'(x) = f(x)$

Indefinite Integral
If $F(x)$ is an antiderivative of $f(x)$ then $\int f(x)dx = F(x) + C$
We call $\int f(x)\,dx$ an indefinite integral.

EXAMPLE 1
Find the following indefinite integrals
(i) $\int \left(x^3 + 4x^2 + 8x - e\right) dx$

(ii) $\int \frac{x^2 + x - 1}{x^3}\,dx$

(iii) $\int \left(\frac{1}{x^2+1} - e^x + 2^x\right) dx$

In the previous section we introduced the definite integral,

Definite Integral
If f is a function defined on a closed interval $[a, b]$, let P be a partition of $[a, b]$ with partition points $x_0, x_1, \ldots x_n$ where

$$a = x_0 < x_1 < \ldots < x_n = b$$

Choose points x_i^* in the subinterval $[x_{i-1}, x_i]$ and let $\Delta x_i = x_i - x_{i-1}$ and $\|P\| = \max\{\Delta x_i\}$. Then the definite integral of f from a to b is

$$\int_a^b f(x)\,dx = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f\left(x_i^*\right) \Delta x_i$$

if this limit exists. If the limit exists for f, then f is integrable on the interval $[a, b]$.

If f is positive on $[a, b]$, then then the definite integral is the area under f and above the x-axis from $x=a$ to $x=b$. If f is not always positive, the definite integral is the net area.

EXAMPLE 2

Evaluate each integral by interpreting it in terms of area.

(i) $\int_1^3 c\,dt$

(ii) $\int_a^x c\,dt$

(iii) $\int_0^x t\,dt$

THE FUNDAMENTAL THEOREM OF CALCULUS, PART 1

If f is continuous on $[a,b]$, then the function g defined by

$$g(x)=\int_a^x f(t)\,dt \qquad a\le x\le b$$

is continuous on $[a,b]$, differentiable on (a,b) and $g'(x)=f(x)$

EXAMPLE 3

Find the derivative of the given functions

(i) $g(x)=\int_{-1}^x \sqrt{t^3+1}\,dt$

(ii) $g(x)=\int_1^{\sqrt{x}} \frac{t^2}{t^2+1}\,dt$

(iii) $g(x)=\int_{x^2}^{\pi} \frac{\sin t}{t}\,dt$

THE FUNDAMENTAL THEOREM OF CALCULUS, PART 2

If f is continuous on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_a^b$$

where F is any antiderivative of f.

EXAMPLE 4

Evaluate the following definite integrals

(i) $\displaystyle\int_0^2 \left(w^3 - 1\right)^2 dw$

(ii) $\displaystyle\int_{-2}^0 \left|x^2 - 1\right| dx$

(iii) $\displaystyle\int_0^{\pi/2} \left(\cos\theta + 2\sin\theta\right) d\theta$

EXAMPLE 5

The velocity function (in meters per second) is given for a particle moving along a line. Find the displacement and distance traveled by the particle during the given time interval.

$v(t) = t^2 - 2t - 8, \quad 1 \le t \le 6$