Lattice paths from Koszul double complexes

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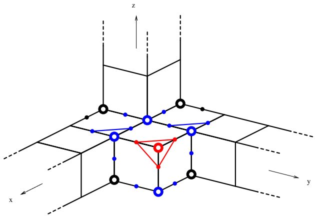
Notation

- $\mathbf{x} = x_1, x_2, \ldots, x_n$
- $S = \Bbbk[\mathbf{x}] = \bigoplus_{\mathbf{b} \in \mathbb{N}^n} \Bbbk \left\{ \mathbf{x}^{\mathbf{b}} \right\}$
- monomial: $\mathbf{x}^{\mathbf{b}} = x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$
  - squarefree monomial: each b_i is either 0 or 1
- I: monomial ideal
- i^{th} Betti number of I in degree $\mathbf{b}$: the rank $\beta_{i,\mathbf{b}}$
  $F_i = \bigoplus_{\mathbf{b} \in \mathbb{N}^n} S(-\mathbf{b})^{\beta_{i,\mathbf{b}}}$ in a minimal free resolution of I

Koszul simplicial complexes



- $K^{\mathbf{b}}I = \{\text{squarefree } \tau \mid \mathbf{x}^{\mathbf{b}-\tau} \in I\}$
- Hochster's formula [Hochster 1977]:

$$\beta_{i,\mathbf{b}}I = \dim_{\Bbbk}\widetilde{H}_{i-1}(K^{\mathbf{b}}I;\Bbbk)$$

- Modules in a minimal free resolution of I:

$$F_i = \bigoplus_{\mathbf{b}\in\mathbb{N}^n} \widetilde{H}_{i-1}(K^{\mathbf{b}}I;\Bbbk) \otimes_{\Bbbk} \Bbbk[\mathbf{x}](-\mathbf{b})$$

- Define a map $F_{i-1} \leftarrow F_i$ by defining a map

$$\widetilde{C}_{i-2}(K^{\mathbf{a}}I) \leftarrow \widetilde{C}_{i-1}(K^{\mathbf{b}}I)$$

that induces a well-defined homomorphism on homology

Goal

- For a monomial ideal $I \subseteq \Bbbk[\mathbf{x}]$, we want to produce vector space homomorphisms

$$\bigoplus_{\mathbf{a} \prec \mathbf{b}} \widetilde{H}_{i-1}(K^{\mathbf{a}} I; \Bbbk) \leftarrow \widetilde{H}_i(K^{\mathbf{b}} I; \Bbbk)$$

whose induced $\Bbbk[\mathbf{x}]$-module homomorphisms

$$\bigoplus_{\mathbf{a} \in \mathbb{N}^n} \widetilde{H}_{i-1}(K^{\mathbf{a}} I; \Bbbk) \otimes_{\Bbbk} \Bbbk[\mathbf{x}](-\mathbf{a}) \leftarrow \bigoplus_{\mathbf{b} \in \mathbb{N}^n} \widetilde{H}_i(K^{\mathbf{b}} I; \Bbbk) \otimes_{\Bbbk} \Bbbk[\mathbf{x}](-\mathbf{b})$$

constitute a free resolution of I.

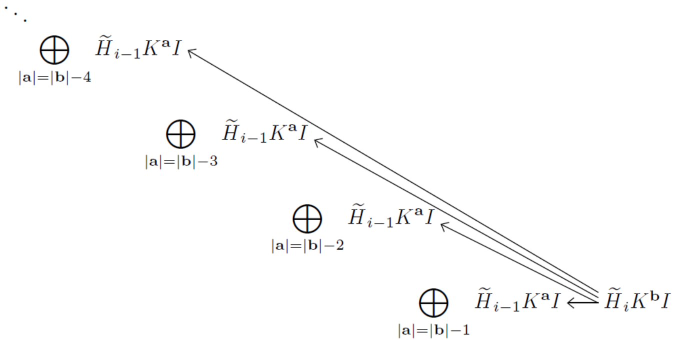
Stratification of homology homomorphisms

- maps $\bigoplus_{\mathbf{a}\preceq\mathbf{b}} \widetilde{H}_{i-1}(K^{\mathbf{a}}I;\Bbbk) \leftarrow \widetilde{H}_i(K^{\mathbf{b}}I;\Bbbk)$ stratify according to lattice distance:

$$\bigoplus_{|\mathbf{a}|=|\mathbf{b}|-j} \widetilde{H}_{i-1}(K^{\mathbf{a}}I;\Bbbk) \leftarrow \widetilde{H}_i(K^{\mathbf{b}}I;\Bbbk) \text{ for } j \geq 1$$

$\bigoplus_{\lvert\mathbf{a}\rvert=0} \widetilde{H}_{-1}K^{\mathbf{a}}I$	$\bigoplus_{\lvert\mathbf{a}\rvert=1} \widetilde{H}_{0}K^{\mathbf{a}}I$	$\bigoplus_{\lvert\mathbf{a}\rvert=2} \widetilde{H}_{1}K^{\mathbf{a}}I$	$\bigoplus_{\lvert\mathbf{a}\rvert=3} \widetilde{H}_{2}K^{\mathbf{a}}I$	$\bigoplus_{\lvert\mathbf{a}\rvert=4} \widetilde{H}_{3}K^{\mathbf{a}}I$	
	$\bigoplus_{\lvert\mathbf{a}\rvert=1} \widetilde{H}_{-1}K^{\mathbf{a}}I$	$\bigoplus_{\lvert\mathbf{a}\rvert=2} \widetilde{H}_{0}K^{\mathbf{a}}I$	$\bigoplus_{\lvert\mathbf{a}\rvert=3} \widetilde{H}_{1}K^{\mathbf{a}}I$	$\bigoplus_{\lvert\mathbf{a}\rvert=4} \widetilde{H}_{2}K^{\mathbf{a}}I$	$\cdots$
		$\bigoplus_{\lvert\mathbf{a}\rvert=2} \widetilde{H}_{-1}K^{\mathbf{a}}I$	$\bigoplus_{\lvert\mathbf{a}\rvert=3} \widetilde{H}_{0}K^{\mathbf{a}}I$	$\bigoplus_{\lvert\mathbf{a}\rvert=4} \widetilde{H}_{1}K^{\mathbf{a}}I$	
		$\ddots$		$\vdots$	

Stratification of homology homomorphisms



Koszul complex notation

- V: n-dimensional $\mathbb{N}^n$-graded $\mathbb{k}$-vector space with basis $z_1, \ldots, z_n$
- Koszul complexes in $\mathbf{x}$ and $\mathbf{y}$:

$$\mathbb{K}_\bullet^{\mathbf{x}} = \bigwedge^\bullet V \otimes \mathbb{k}[\mathbf{x}] \quad \text{and} \quad \mathbb{K}_\bullet^{\mathbf{y}} = \bigwedge^\bullet V \otimes \mathbb{k}[\mathbf{y}]$$

with $\mathbb{N}^n$-graded differentials

$$\mathbf{z}^\sigma \otimes \mathbf{y}^{\mathbf{b}-\sigma} \mapsto \sum_{j \in \sigma} \pm \mathbf{z}^{\sigma - \mathbf{e}_j} \otimes \mathbf{y}^{\mathbf{b}+\mathbf{e}_j-\sigma}$$

- Koszul complex of $I^{\mathbf{y}}$: $\bigwedge^\bullet V \otimes \mathbb{k}[\mathbf{y}] \otimes_{\mathbb{k}[\mathbf{y}]} I^{\mathbf{y}}$ with differential induced by the differential of the Koszul complex in $\mathbf{y}$

$$\bigwedge^\bullet V \otimes \mathbb{k}[\mathbf{x}] \otimes \mathbb{k}[\mathbf{y}] \otimes_{\mathbb{k}[\mathbf{y}]} I^{\mathbf{y}}$$

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$$\mathbb{K}^{\mathbf{y}}_{\bullet}(I) \otimes \Bbbk[\mathbf{x}] \;=\; \bigwedge^{\bullet} V \otimes \Bbbk[\mathbf{x}] \otimes \Bbbk[\mathbf{y}] \otimes_{\Bbbk[\mathbf{y}]} I^{\mathbf{y}}$$

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$$\begin{array}{c} \bigwedge^\bullet V \otimes \Bbbk[\mathbf{x}] \otimes \Bbbk[\mathbf{y}] \otimes_{\Bbbk[\mathbf{y}]} I^{\mathbf{y}} \\ \parallel \\ \mathbb{K}_\bullet^{\mathbf{x}+\mathbf{y}}(I) \end{array}$$

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$$\mathbb{K}_\bullet^{\mathbf{y}}(I) \otimes \Bbbk[\mathbf{x}] \;=\; \bigwedge^\bullet V \otimes \Bbbk[\mathbf{x}] \otimes \Bbbk[\mathbf{y}] \otimes_{\Bbbk[\mathbf{y}]} I^{\mathbf{y}} \;=\; \mathbb{K}_\bullet^{\mathbf{x}} \otimes \Bbbk[\mathbf{y}] \otimes_{\Bbbk[\mathbf{y}]} I^{\mathbf{y}}$$

$$\| $$

$$\mathbb{K}_\bullet^{\mathbf{x}+\mathbf{y}}(I)$$

Koszul bicomplexes

$$\mathbb{K}^{\mathbf{y}}_{\bullet}(I) \otimes \Bbbk[\mathbf{x}] \;=\; \bigwedge\nolimits^{\bullet} V \otimes \Bbbk[\mathbf{x}] \otimes \Bbbk[\mathbf{y}] \otimes_{\Bbbk[\mathbf{y}]} I^{\mathbf{y}}$$

$$\begin{array}{c|cccc}
\hline
& \displaystyle\bigoplus_{|\mathbf{a}|=0} \mathbb{K}^{\mathbf{y}}_0 \otimes I^{\mathbf{y}}_{\mathbf{a}} \otimes \Bbbk[\mathbf{x}] & \displaystyle\bigoplus_{|\mathbf{a}|=0} \mathbb{K}^{\mathbf{y}}_1 \otimes I^{\mathbf{y}}_{\mathbf{a}} \otimes \Bbbk[\mathbf{x}] & \displaystyle\bigoplus_{|\mathbf{a}|=0} \mathbb{K}^{\mathbf{y}}_2 \otimes I^{\mathbf{y}}_{\mathbf{a}} \otimes \Bbbk[\mathbf{x}] & \displaystyle\bigoplus_{|\mathbf{a}|=0} \mathbb{K}^{\mathbf{y}}_3 \otimes I^{\mathbf{y}}_{\mathbf{a}} \otimes \Bbbk[\mathbf{x}] \\
& & \downarrow & \downarrow & \downarrow \\
& & \displaystyle\bigoplus_{|\mathbf{a}|=1} \mathbb{K}^{\mathbf{y}}_0 \otimes I^{\mathbf{y}}_{\mathbf{a}} \otimes \Bbbk[\mathbf{x}] & \displaystyle\bigoplus_{|\mathbf{a}|=1} \mathbb{K}^{\mathbf{y}}_1 \otimes I^{\mathbf{y}}_{\mathbf{a}} \otimes \Bbbk[\mathbf{x}] & \displaystyle\bigoplus_{|\mathbf{a}|=1} \mathbb{K}^{\mathbf{y}}_2 \otimes I^{\mathbf{y}}_{\mathbf{a}} \otimes \Bbbk[\mathbf{x}] \\
& & & \downarrow & \downarrow \\
& & & \displaystyle\bigoplus_{|\mathbf{a}|=2} \mathbb{K}^{\mathbf{y}}_0 \otimes I^{\mathbf{y}}_{\mathbf{a}} \otimes \Bbbk[\mathbf{x}] & \displaystyle\bigoplus_{|\mathbf{a}|=2} \mathbb{K}^{\mathbf{y}}_1 \otimes I^{\mathbf{y}}_{\mathbf{a}} \otimes \Bbbk[\mathbf{x}] \\
& & & & \downarrow \\
& & & \ddots & \vdots
\end{array}$$

Koszul bicomplexes

$$\bigwedge^{\bullet} V \otimes \Bbbk[\mathbf{x}] \otimes \Bbbk[\mathbf{y}] \otimes_{\Bbbk[\mathbf{y}]} I^{\mathbf{y}} \quad = \quad \mathbb{K}^{\mathbf{x}}_{\bullet} \otimes \Bbbk[\mathbf{y}] \otimes_{\Bbbk[\mathbf{y}]} I^{\mathbf{y}}$$

$$\begin{array}{|ccccccc}
\hline
\bigoplus_{|\mathbf{a}|=0} \mathbb{K}^{\mathbf{y}}_0 \otimes I^{\mathbf{y}}_{\mathbf{a}} \otimes \Bbbk[\mathbf{x}] & \leftarrow & \bigoplus_{|\mathbf{a}|=0} \mathbb{K}^{\mathbf{y}}_1 \otimes I^{\mathbf{y}}_{\mathbf{a}} \otimes \Bbbk[\mathbf{x}] & \leftarrow & \bigoplus_{|\mathbf{a}|=0} \mathbb{K}^{\mathbf{y}}_2 \otimes I^{\mathbf{y}}_{\mathbf{a}} \otimes \Bbbk[\mathbf{x}] & \leftarrow & \bigoplus_{|\mathbf{a}|=0} \mathbb{K}^{\mathbf{y}}_3 \otimes I^{\mathbf{y}}_{\mathbf{a}} \otimes \Bbbk[\mathbf{x}] \\
 & & \bigoplus_{|\mathbf{a}|=1} \mathbb{K}^{\mathbf{y}}_0 \otimes I^{\mathbf{y}}_{\mathbf{a}} \otimes \Bbbk[\mathbf{x}] & \leftarrow & \bigoplus_{|\mathbf{a}|=1} \mathbb{K}^{\mathbf{y}}_1 \otimes I^{\mathbf{y}}_{\mathbf{a}} \otimes \Bbbk[\mathbf{x}] & \leftarrow & \bigoplus_{|\mathbf{a}|=1} \mathbb{K}^{\mathbf{y}}_2 \otimes I^{\mathbf{y}}_{\mathbf{a}} \otimes \Bbbk[\mathbf{x}] \\
 & & & & \bigoplus_{|\mathbf{a}|=2} \mathbb{K}^{\mathbf{y}}_0 \otimes I^{\mathbf{y}}_{\mathbf{a}} \otimes \Bbbk[\mathbf{x}] & \leftarrow & \bigoplus_{|\mathbf{a}|=2} \mathbb{K}^{\mathbf{y}}_1 \otimes I^{\mathbf{y}}_{\mathbf{a}} \otimes \Bbbk[\mathbf{x}] \\
 & & & & & \ddots & \vdots
\end{array}$$

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$$\bigwedge\nolimits^{\bullet} V \otimes \mathbb{k}[\mathbf{x}] \otimes \mathbb{k}[\mathbf{y}] \otimes_{\mathbb{k}[\mathbf{y}]} I^{\mathbf{y}}$$

$$\|$$

$$\mathbb{K}_{\bullet}^{\mathbf{x}+\mathbf{y}}(I)$$

$$\begin{array}{c|ccccccc}
\hline
& \bigoplus\limits_{|\mathbf{a}|=0} \mathbb{K}_0^{\mathbf{y}} \otimes I_{\mathbf{a}}^{\mathbf{y}} \otimes \mathbb{k}[\mathbf{x}] & \leftarrow & \bigoplus\limits_{|\mathbf{a}|=0} \mathbb{K}_1^{\mathbf{y}} \otimes I_{\mathbf{a}}^{\mathbf{y}} \otimes \mathbb{k}[\mathbf{x}] & \leftarrow & \bigoplus\limits_{|\mathbf{a}|=0} \mathbb{K}_2^{\mathbf{y}} \otimes I_{\mathbf{a}}^{\mathbf{y}} \otimes \mathbb{k}[\mathbf{x}] & \leftarrow & \bigoplus\limits_{|\mathbf{a}|=0} \mathbb{K}_3^{\mathbf{y}} \otimes I_{\mathbf{a}}^{\mathbf{y}} \otimes \mathbb{k}[\mathbf{x}] \\
& & & \downarrow & & \downarrow & & \downarrow \\
& & & \bigoplus\limits_{|\mathbf{a}|=1} \mathbb{K}_0^{\mathbf{y}} \otimes I_{\mathbf{a}}^{\mathbf{y}} \otimes \mathbb{k}[\mathbf{x}] & \leftarrow & \bigoplus\limits_{|\mathbf{a}|=1} \mathbb{K}_1^{\mathbf{y}} \otimes I_{\mathbf{a}}^{\mathbf{y}} \otimes \mathbb{k}[\mathbf{x}] & \leftarrow & \bigoplus\limits_{|\mathbf{a}|=1} \mathbb{K}_2^{\mathbf{y}} \otimes I_{\mathbf{a}}^{\mathbf{y}} \otimes \mathbb{k}[\mathbf{x}] \\
& & & & & \downarrow & & \downarrow \\
& & & & & \bigoplus\limits_{|\mathbf{a}|=2} \mathbb{K}_0^{\mathbf{y}} \otimes I_{\mathbf{a}}^{\mathbf{y}} \otimes \mathbb{k}[\mathbf{x}] & \leftarrow & \bigoplus\limits_{|\mathbf{a}|=2} \mathbb{K}_1^{\mathbf{y}} \otimes I_{\mathbf{a}}^{\mathbf{y}} \otimes \mathbb{k}[\mathbf{x}] \\
& & & & & & & \downarrow \\
& & & & & & \ddots & \vdots
\end{array}$$

Koszul bicomplexes

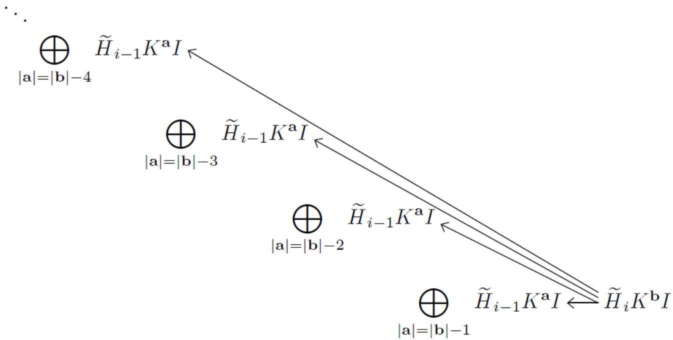
$$\left(\mathbb{K}^{\mathbf{y}}_{\bullet}(I^{\mathbf{y}})\right)_{\mathbf{b}} \cong \widetilde{C}_{\bullet} K^{\mathbf{b}} I$$

$$\begin{array}{|cccc}
\hline
\bigoplus\limits_{|\mathbf{a}|=0} \mathbb{K}^{\mathbf{y}}_0 \otimes I^{\mathbf{y}}_{\mathbf{a}} \otimes \Bbbk[\mathbf{x}] & \bigoplus\limits_{|\mathbf{a}|=0} \mathbb{K}^{\mathbf{y}}_1 \otimes I^{\mathbf{y}}_{\mathbf{a}} \otimes \Bbbk[\mathbf{x}] & \bigoplus\limits_{|\mathbf{a}|=0} \mathbb{K}^{\mathbf{y}}_2 \otimes I^{\mathbf{y}}_{\mathbf{a}} \otimes \Bbbk[\mathbf{x}] & \bigoplus\limits_{|\mathbf{a}|=0} \mathbb{K}^{\mathbf{y}}_3 \otimes I^{\mathbf{y}}_{\mathbf{a}} \otimes \Bbbk[\mathbf{x}] \\
 & \downarrow & \downarrow & \downarrow \\
 & \bigoplus\limits_{|\mathbf{a}|=1} \mathbb{K}^{\mathbf{y}}_0 \otimes I^{\mathbf{y}}_{\mathbf{a}} \otimes \Bbbk[\mathbf{x}] & \bigoplus\limits_{|\mathbf{a}|=1} \mathbb{K}^{\mathbf{y}}_1 \otimes I^{\mathbf{y}}_{\mathbf{a}} \otimes \Bbbk[\mathbf{x}] & \bigoplus\limits_{|\mathbf{a}|=1} \mathbb{K}^{\mathbf{y}}_2 \otimes I^{\mathbf{y}}_{\mathbf{a}} \otimes \Bbbk[\mathbf{x}] \\
 & & \downarrow & \downarrow \\
 & & \bigoplus\limits_{|\mathbf{a}|=2} \mathbb{K}^{\mathbf{y}}_0 \otimes I^{\mathbf{y}}_{\mathbf{a}} \otimes \Bbbk[\mathbf{x}] & \bigoplus\limits_{|\mathbf{a}|=2} \mathbb{K}^{\mathbf{y}}_1 \otimes I^{\mathbf{y}}_{\mathbf{a}} \otimes \Bbbk[\mathbf{x}] \\
 & & & \downarrow \\
 & & \ddots & \vdots
\end{array}$$

Koszul bicomplexes

$\bigoplus_{\lvert\mathbf{a}\rvert=0} \widetilde{H}_{-1} K^{\mathbf{a}} I \otimes \Bbbk[\mathbf{x}]$	$\bigoplus_{\lvert\mathbf{a}\rvert=1} \widetilde{H}_{0} K^{\mathbf{a}} I \otimes \Bbbk[\mathbf{x}]$	$\bigoplus_{\lvert\mathbf{a}\rvert=2} \widetilde{H}_{1} K^{\mathbf{a}} I \otimes \Bbbk[\mathbf{x}]$	$\bigoplus_{\lvert\mathbf{a}\rvert=3} \widetilde{H}_{2} K^{\mathbf{a}} I \otimes \Bbbk[\mathbf{x}]$
	$\bigoplus_{\lvert\mathbf{a}\rvert=1} \widetilde{H}_{-1} K^{\mathbf{a}} I \otimes \Bbbk[\mathbf{x}]$	$\bigoplus_{\lvert\mathbf{a}\rvert=2} \widetilde{H}_{0} K^{\mathbf{a}} I \otimes \Bbbk[\mathbf{x}]$	$\bigoplus_{\lvert\mathbf{a}\rvert=3} \widetilde{H}_{1} K^{\mathbf{a}} I \otimes \Bbbk[\mathbf{x}]$
		$\bigoplus_{\lvert\mathbf{a}\rvert=2} \widetilde{H}_{-1} K^{\mathbf{a}} I \otimes \Bbbk[\mathbf{x}]$	$\bigoplus_{\lvert\mathbf{a}\rvert=3} \widetilde{H}_{0} K^{\mathbf{a}} I \otimes \Bbbk[\mathbf{x}]$
		$\ddots$	$\vdots$

Koszul bicomplexes



Wall complexes

- $C_{\bullet\bullet}$: bicomplex of R-modules
  - vertical differential $d = d_0$
  - horizontal differential d_1
- a **vertical splitting** of $C_{\bullet\bullet}$ consists of a differential

$$d^+ = d^+_{pq} : C_{pq} \to C_{p,q+1}$$

  such that $dd^+d = d$ and $d^+dd^+ = d^+$
- equivalent to a direct sum decomposition

$$C_{pq} = B'_{p,q-1} \oplus H_{pq} \oplus B_{pq}$$

- In the Koszul bicomplex $\mathbb{K}_{\bullet\bullet}(I)$, the vertical differential is induced by the boundary map $\partial^{\mathbf{b}}$ of $K^{\mathbf{b}}I$.

Wall complexes

Fix a vertical splitting of $C_{\bullet\bullet}$

1. $(\mathbb{1} - d^+ d) : C_{pq} \to Z_{pq}$ and $(\mathbb{1} - dd^+) : Z_{pq} \to H_{pq}$ are projections onto Z_{pq} and H_{pq}, respectively.
2. The composite of the upward and leftward differentials induces homomorphisms

$$\begin{array}{ccc} C_{p-1,q+1} & & \\ & \nwarrow^{d^+ d_1} & \\ & & C_{pq} \end{array}$$

Together, these homomorphisms induce morphisms

$$\omega_j : \widetilde{H}_{pq} \to \widetilde{H}_{p-j,q+j-1} \text{ for } j \geq 1 \text{ via}$$

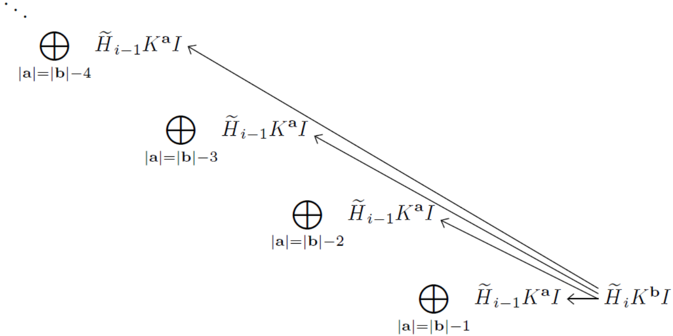
$$\omega_j = (\mathbb{1} - d^+ d)(d_1 d^+)^{j-1} d_1 (\mathbb{1} - dd^+).$$

- $\widetilde{H}_{\bullet\bullet}$ is the natural Wall complex of $C_{\bullet\bullet}$ with the differentials $\omega_0 = 0$ and ω_j as above

Wall complexes

$$\omega_j : \widetilde{H}_{pq} \to \widetilde{H}_{p-j,q+j-1} \text{ for } j \geq 1 \text{ via}$$

$$\omega_j = (\mathbb{1} - d^+ d)(d_1 d^+)^{j-1} d_1 (\mathbb{1} - dd^+).$$



Monomial resolutions from splittings

Theorem (Eagon–Miller–O. 2019)

Any splittings $\partial^{\mathbf{b}+}$ of the differentials $\partial^{\mathbf{b}}$ of $K^{\mathbf{b}}I$ yield a free resolution of I whose differentials are induced by

$$\widetilde{H}_{i-1}K^{\mathbf{a}}I \xleftarrow{D} \widetilde{H}_i K^{\mathbf{b}}I$$

that acts on any i-cycle in $\widetilde{Z}_i K^{\mathbf{b}}I$ via

$$D = \sum_{\lambda \in \Lambda(\mathbf{a},\mathbf{b})} (\mathbb{1}^{\mathbf{a}} - \partial_i^{\mathbf{a}+}\partial_i^{\mathbf{a}}) d_1^{\lambda_\ell} \left(\prod_{j=1}^{\ell-1} \partial_i^{\mathbf{b}_j+} d_1^{\lambda_j} \right) (\mathbb{1}^{\mathbf{b}} - \partial_{i+1}^{\mathbf{b}}\partial_{i+1}^{\mathbf{b}+}).$$

Wall differentials yield lattice paths

$$
\begin{array}{ccccccccc}
\widetilde{C}_{i-1}K^{\mathbf{a}}I & \xleftarrow{d_1^{\lambda_\ell}} & & & & & & & \\
\partial_i^{\mathbf{a}+}\uparrow\ \downarrow\partial_i^{\mathbf{a}} & & \uparrow \partial_i^{\mathbf{b}_{\ell-1}+} & & & & & & \\
 & & \widetilde{C}_{i-1}K^{\mathbf{b}_{\ell-1}}I & \xleftarrow{d_1^{\lambda_{\ell-1}}} & & & & & \\
 & & & \ddots & \uparrow \partial_i^{\mathbf{b}_2+} & & & & \\
 & & & & \widetilde{C}_{i-1}K^{\mathbf{b}_2}I & \xleftarrow{d_1^{\lambda_2}} & & & \\
 & & & & & & \uparrow \partial_i^{\mathbf{b}_1+} & & \partial_{i+1}^{\mathbf{b}}\downarrow\ \uparrow\partial_{i+1}^{\mathbf{b}+} \\
 & & & & & & \widetilde{C}_{i-1}K^{\mathbf{b}_1}I & \xleftarrow{d_1^{\lambda_1}} & \widetilde{C}_iK^{\mathbf{b}}I
\end{array}
$$

- $\mathbb{K}_\bullet^{\mathbf{y}}(I^{\mathbf{y}})_{\mathbf{b}} \cong \widetilde{C}_\bullet K^{\mathbf{b}}I$
- $\mathbf{z}^\tau \otimes \mathbf{y}^{\mathbf{b}-\tau}$ identified with $\tau \in K^{\mathbf{b}}I$
- d_1 acts as the boundary operator and can be decomposed as $d_1 = d_1^{e_1} + \cdots + d_1^{e_n}$, where $d_1^{e_k}(\mathbf{z}^\tau \otimes \mathbf{y}^{\mathbf{b}-\tau}) = \pm\mathbf{z}^{\tau-e_k} \otimes \mathbf{y}^{\mathbf{b}-\tau}$, identified with $\tau - e_k \in K^{\mathbf{b}-e_k}I$

Wall differentials yield lattice paths

$$
\begin{array}{ccccccc}
\widetilde{C}_{i-1}K^{\mathbf{a}}I & \xleftarrow{d_1^{\lambda_\ell}} & & & & & \\
\partial_i^{\mathbf{a}+}\uparrow\downarrow\partial_i^{\mathbf{a}} & & \uparrow\partial_i^{\mathbf{b}_{\ell-1}+} & & & & \\
 & & \widetilde{C}_{i-1}K^{\mathbf{b}_{\ell-1}}I & \xleftarrow{d_1^{\lambda_{\ell-1}}} & & & \\
 & & & \ddots & \uparrow\partial_i^{\mathbf{b}_2+} & & \\
 & & & & \widetilde{C}_{i-1}K^{\mathbf{b}_2}I & \xleftarrow{d_1^{\lambda_2}} & \\
 & & & & & \uparrow\partial_i^{\mathbf{b}_1+} & \partial_{i+1}^{\mathbf{b}}\downarrow\uparrow\partial_{i+1}^{\mathbf{b}+} \\
 & & & & & \widetilde{C}_{i-1}K^{\mathbf{b}_1}I \xleftarrow{d_1^{\lambda_1}} & \widetilde{C}_iK^{\mathbf{b}}I
\end{array}
$$

- $\mathbb{K}_\bullet^{\mathbf{y}}(I^{\mathbf{y}})_{\mathbf{b}} \cong \widetilde{C}_\bullet K^{\mathbf{b}}I$
- $\partial_i^{\mathbf{b}}$ acts as the boundary operator in $K^{\mathbf{b}}I$
- $\partial_i^{\mathbf{b}+}(\tau)$ is a linear combination of i-dimensional faces in $K^{\mathbf{b}}I$

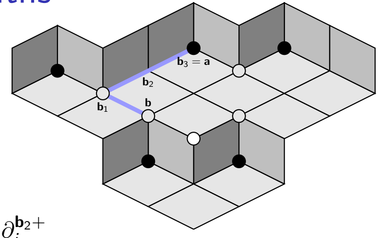
Wall differentials yield lattice paths

$$\widetilde{C}_{i-1}K^{\mathbf{a}}I \xleftarrow{d_1^{\lambda_\ell}} \qquad \partial_i^{\mathbf{a}+}\uparrow \downarrow \partial_i^{\mathbf{a}} \qquad \uparrow \partial_i^{\mathbf{b}_{\ell-1}+}$$

$$\widetilde{C}_{i-1}K^{\mathbf{b}_{\ell-1}}I \xleftarrow{d_1^{\lambda_{\ell-1}}} \quad \ddots \quad \uparrow \partial_i^{\mathbf{b}_2+}$$

$$\widetilde{C}_{i-1}K^{\mathbf{b}_2}I \xleftarrow{d_1^{\lambda_2}} \qquad \uparrow \partial_i^{\mathbf{b}_1+} \qquad \partial_{i+1}^{\mathbf{b}} \downarrow \uparrow \partial_{i+1}^{\mathbf{b}+}$$

$$\widetilde{C}_{i-1}K^{\mathbf{b}_1}I \xleftarrow{d_1^{\lambda_1}} \widetilde{C}_i K^{\mathbf{b}}I$$



- lattice path $\lambda = (\mathbf{a} = \mathbf{b}_\ell, \mathbf{b}_{\ell-1}, \ldots, \mathbf{b}_0 = \mathbf{b})$ identified with $(\lambda_\ell, \lambda_{\ell-1}, \ldots, \lambda_1)$, where $\lambda_i = \mathbf{b}_{i-1} - \mathbf{b}_i$
- $D = \sum_{\lambda \in \Lambda(\mathbf{a},\mathbf{b})} (\mathbb{1}^{\mathbf{a}} - \partial_i^{\mathbf{a}+}\partial_i^{\mathbf{a}}) d_1^{\lambda_\ell} \left(\prod_{j=1}^{\ell-1} \partial_i^{\mathbf{b}_j+} d_1^{\lambda_j} \right) (\mathbb{1}^{\mathbf{b}} - \partial_{i+1}^{\mathbf{b}} \partial_{i+1}^{\mathbf{b}+})$

Chain-link fences

$$
\begin{array}{ccccccccc}
\widetilde{C}_{i-1}K^{\mathbf{a}}I & \xleftarrow{d_1^{\lambda_\ell}} & & & & & & & \\
\partial_i^{\mathbf{a}+}\uparrow\ \downarrow\partial_i^{\mathbf{a}} & & \uparrow \partial_i^{\mathbf{b}_{\ell-1}+} & & & & & & \\
& & \widetilde{C}_{i-1}K^{\mathbf{b}_{\ell-1}}I & \xleftarrow{d_1^{\lambda_{\ell-1}}} & & & & & \\
& & & \ddots & \uparrow \partial_i^{\mathbf{b}_2+} & & & & \\
& & & & \widetilde{C}_{i-1}K^{\mathbf{b}_2}I & \xleftarrow{d_1^{\lambda_2}} & & & \\
& & & & & & \uparrow \partial_i^{\mathbf{b}_1+} & & \partial_{i+1}^{\mathbf{b}}\downarrow\ \uparrow\partial_{i+1}^{\mathbf{b}+} \\
& & & & & & \widetilde{C}_{i-1}K^{\mathbf{b}_1}I & \xleftarrow{d_1^{\lambda_1}} & \widetilde{C}_iK^{\mathbf{b}}I
\end{array}
$$

- gives chain-link fences:

$$
\begin{array}{ccccccccccccc}
 & & \tau_{\ell-1} & & & \cdots & & & \tau_1 & & \tau_0 & — & \tau \\
 & / & & \backslash & / & & \backslash & / & & \backslash\ / & & & \\
\sigma — \sigma_\ell & & & \sigma_{\ell-1} & & & \sigma_2 & & \sigma_1 & & & &
\end{array}
$$

in which $\tau_j \in K_i^{\mathbf{b}_j}I$ and $\sigma_j \in K_{i-1}^{\mathbf{b}_j}I$

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