Minimal resolutions of monomial ideals

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Joint work with John Eagon and Ezra Miller

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Notation and definitions

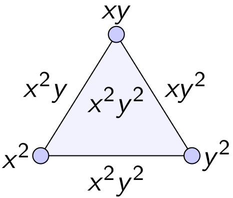
- $\mathbf{x} = x_1, x_2, \ldots, x_n$
- $S = \Bbbk[\mathbf{x}] = \bigoplus_{\mathbf{b} \in \mathbb{N}^n} \Bbbk \left\{ \mathbf{x}^{\mathbf{b}} \right\}$
- monomial: $\mathbf{x}^{\mathbf{b}} = x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$
  - squarefree monomial: each b_i is either 0 or 1
- I: monomial ideal
- free S-module of rank r: direct sum S^r of copies of S
- a free resolution of I: a complex of free modules

$$\mathcal{F}_\bullet : 0 \leftarrow F_0 \xleftarrow{\varphi_1} F_1 \leftarrow \cdots \leftarrow F_{r-1} \xleftarrow{\varphi_r} F_r \leftarrow 0$$

that is exact everywhere except in homological degree 0, where $I = F_0/\mathrm{im}(\varphi_1)$

Example: The Taylor resolution

- The homomorphisms for the Taylor resolution are supported on the full $(r-1)$-dimensional simplex whose vertices are labeled by the r generators of I and whose faces are labeled by the least common multiples of the vertices making up each face.
- Example: $I = \langle x^2, y^2, xy \rangle$



$$
\begin{array}{c} \\ x^2 \\ y^2 \\ xy \end{array}
\begin{array}{c} \begin{array}{ccc} x^2y & x^2y^2 & xy^2 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ -1 & 0 & 1 \end{pmatrix} \end{array}
\qquad
\begin{array}{c} \\ x^2y \\ x^2y^2 \\ xy^2 \end{array}
\begin{array}{c} x^2y^2 \\ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \end{array}
$$

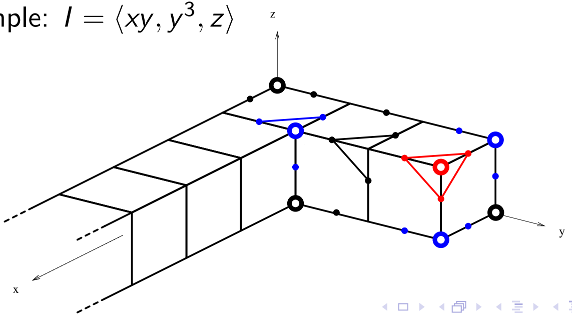
$$0 \leftarrow S^3 \longleftarrow S^3 \longleftarrow S \leftarrow 0$$

Betti numbers and the Koszul simplicial complex

- i^{th} Betti number of I in degree $\mathbf{b}$: the rank $\beta_{i,\mathbf{b}}$ of $F_i = \bigoplus_{\mathbf{b}\in\mathbb{N}^n} S(-\mathbf{b})^{\beta_{i,\mathbf{b}}}$ in a minimal free resolution of I
- Koszul simplicial complex: $K^{\mathbf{b}}I = \{\tau \in \{0,1\}^n \mid \mathbf{x}^{\mathbf{b}-\tau} \in I\}$
- **Hochster's formula** ([Hoc77], see [MS05]): Given a degree vector $\mathbf{b} \in \mathbb{N}^n$, the Betti numbers of I in degree $\mathbf{b}$ are

$$\beta_{i,\mathbf{b}}(I) = \dim_{\mathbb{k}} \mathrm{Tor}_i^S(\mathbb{k}, I)_{\mathbf{b}} = \dim_{\mathbb{k}} \widetilde{H}_{i-1}(K^{\mathbf{b}}I; \mathbb{k}).$$

- $F_i = \bigoplus_{\mathbf{b}\in\mathbb{N}^n} \widetilde{H}_{i-1}(K^{\mathbf{b}}I) \otimes \mathbb{k}[\mathbf{x}](-\mathbf{b})$
- Example: $I = \langle xy, y^3, z\rangle$



Goal

- Produce a free resolution of I that is minimal, universal, canonical, closed-form, and combinatorial.
- To do this, we must produce vector space homomorphisms

$$\bigoplus_{\mathbf{a} \preceq \mathbf{b}} \widetilde{H}_{i-1}(K^{\mathbf{a}} I; \mathbb{k}) \leftarrow \widetilde{H}_i(K^{\mathbf{b}} I; \mathbb{k})$$

whose induced $\mathbb{k}[\mathbf{x}]$-module homomorphisms

$$\bigoplus_{\mathbf{a} \in \mathbb{N}^n} \widetilde{H}_{i-1}(K^{\mathbf{a}} I; \mathbb{k}) \otimes_{\mathbb{k}} \mathbb{k}[\mathbf{x}](-\mathbf{a}) \leftarrow \bigoplus_{\mathbf{b} \in \mathbb{N}^n} \widetilde{H}_i(K^{\mathbf{b}} I; \mathbb{k}) \otimes_{\mathbb{k}} \mathbb{k}[\mathbf{x}](-\mathbf{b})$$

give a free resolution of I.

Past constructions

- Taylor resolution (not minimal) [Tay66]
- Lyubeznik resolutions (not minimal or canonical) [Lyu88]
- Eliahou–Kervaire resolution of stable ideals (not universal) [EK90]
- Wall resolutions (not proved combinatorial or universal) [Eag90]
- Scarf complex (minimal for generic ideals) [Sca86, BPS98]
- Hull resolution (minimal for generic ideals) [BS98]
- Minimal Taylor resolutions (combinatorial algorithm, not closed-form or canonical) [Yuz99]
- Planar graph resolutions for trivariate monomial ideals (not canonical or universal) [Mil02]
- Buchberger resolutions (not canonical or minimal) [MS05,OW16]
- Subsequent development: algorithmic canonical resolutions (combinatorial algorithm, not closed-form) [Tch19]

Sylvan matrices

- Problem: homology vector spaces do not have canonical bases
- Solution: specify a linear map $\widetilde{C}_{i-1}K^{\mathbf{a}}I \leftarrow \widetilde{C}_i K^{\mathbf{b}}I$ using their natural bases (faces of dimensions i and $i-1$) that induces a well-defined homomorphism $\widetilde{H}_{i-1}K^{\mathbf{a}}I \leftarrow \widetilde{H}_i K^{\mathbf{b}}I$

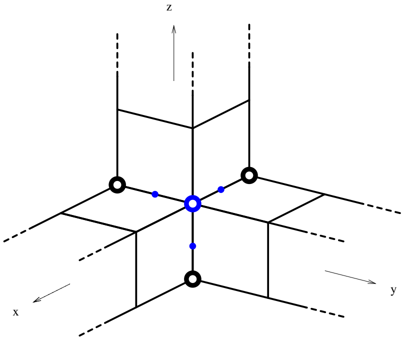
$$
\begin{array}{c}
\qquad\qquad \begin{array}{ccc} \tau_1 & \cdots & \tau_n \end{array} \\
\begin{array}{c} \sigma_1 \\ \vdots \\ \sigma_m \end{array}
\left[\begin{array}{ccc} & & \\ & & \\ & & \end{array} \right]
\end{array}
$$

$$
\widetilde{H}_{i-1}K^{\mathbf{a}} \otimes \langle \mathbf{x}^{\mathbf{a}} \rangle \longleftarrow \widetilde{H}_i K^{\mathbf{b}} \otimes \langle \mathbf{x}^{\mathbf{b}} \rangle
$$

- τ_j is an i-face for all j
- σ_j is an $(i-1)$-face for all j

Example: $I = \langle xy, xz, yz \rangle$

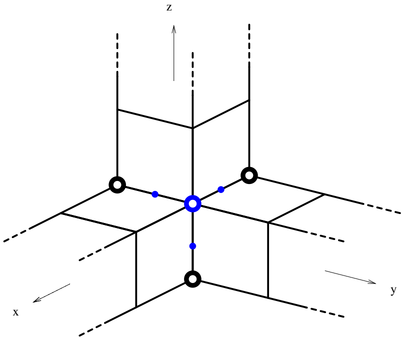
$\beta_{1,111}(I) = 2$



$$\begin{matrix} \widetilde{H}_{-1} K^{110} \otimes \langle xy \rangle \\ \oplus \\ \widetilde{H}_{-1} K^{101} \otimes \langle xz \rangle \\ \oplus \\ \widetilde{H}_{-1} K^{011} \otimes \langle yz \rangle \end{matrix} \xleftarrow{\begin{matrix} & x & y & z \\ \varnothing & [0 & 0 & 1] \\ \varnothing & [0 & 1 & 0] \\ \varnothing & [1 & 0 & 0] \end{matrix}} \widetilde{H}_0 K^{111} \otimes \langle xyz \rangle$$

Example: $I = \langle xy, xz, yz \rangle$

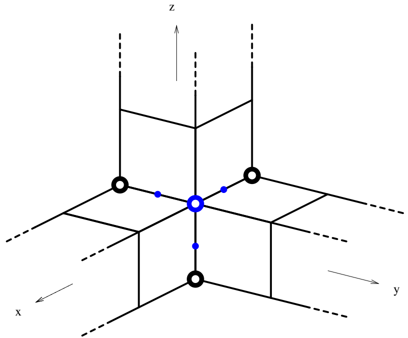
$\beta_{1,111}(I) = 2$



$$
\begin{array}{ccc}
\widetilde{H}_{-1} K^{110} \otimes \langle xy \rangle & & \\
\oplus & \begin{array}{c} \quad x \quad y \quad z \\ \begin{array}{c} \varnothing \\ \varnothing \\ \varnothing \end{array} \begin{bmatrix} [0 & 0 & 1] \\ [0 & 1 & 0] \\ [1 & 0 & 0] \end{bmatrix} \end{array} & (x - z) \otimes xyz \\
\widetilde{H}_{-1} K^{101} \otimes \langle xz \rangle & \longleftarrow & \widetilde{H}_{0} K^{111} \otimes \langle xyz \rangle \\
\oplus & & \\
\widetilde{H}_{-1} K^{011} \otimes \langle yz \rangle & &
\end{array}
$$

Example: $I = \langle xy, xz, yz \rangle$

$\beta_{1,111}(I) = 2$



$$
\begin{array}{cccc}
-\varnothing \otimes z \cdot xy & \widetilde{H}_{-1}K^{110} \otimes \langle xy \rangle & & (x - z) \otimes xyz \\
& \oplus & \begin{array}{c} \quad x \quad y \quad z \\ \begin{array}{c} \varnothing \\ \varnothing \\ \varnothing \end{array} \begin{bmatrix} [0 & 0 & 1] \\ [0 & 1 & 0] \\ [1 & 0 & 0] \end{bmatrix} \end{array} & \\
& \widetilde{H}_{-1}K^{101} \otimes \langle xz \rangle & \longleftarrow & \widetilde{H}_0 K^{111} \otimes \langle xyz \rangle \\
& \oplus & & \\
\varnothing \otimes x \cdot yz & \widetilde{H}_{-1}K^{011} \otimes \langle yz \rangle & &
\end{array}
$$

Canonical differential

Theorem (Eagon-Miller-O.)

In characteristic 0 *and almost all positive characteristics, there is a* **canonical sylvan homomorphism**

$$\widetilde{C}_{i-1}K^{\mathbf{a}}I \xleftarrow{D^{\mathbf{ab}}} \widetilde{C}_i K^{\mathbf{b}}I$$

such that $D(\widetilde{Z}_i K^{\mathbf{b}} I) \subseteq \widetilde{Z}_{i-1} K^{\mathbf{a}} I$ *and* $D(\widetilde{B}_i K^{\mathbf{b}} I) = 0$*, explicitly given by its* **sylvan matrix** *with combinatorial entries*

$$D_{\sigma\tau} = \sum_{\lambda \in \Lambda(\mathbf{a},\mathbf{b})} \frac{1}{\Delta_{i,\lambda} I} \sum_{\varphi \in \Phi_{\sigma\tau}(\lambda)} w_\varphi,$$

where

- $\Lambda(\mathbf{a},\mathbf{b}) = \{$*saturated, decreasing lattice paths from* $\mathbf{b}$ *to* $\mathbf{a}\}$,
- $\Phi_{\sigma\tau}(\lambda) = \{$*chain-link fences from* τ *to* σ *along* $\lambda\}$, *and*
- $w_\varphi =$ *weight of* φ.

Chain-link fences

- Fix a lattice path $\lambda = (\mathbf{a} = \mathbf{b}_\ell, \mathbf{b}_{\ell-1}, \ldots, \mathbf{b}_1, \mathbf{b}_0 = \mathbf{b})$. A chain-link fence φ from an i-simplex τ to an $(i-1)$-simplex σ along λ is a sequence

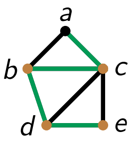
$$\begin{array}{ccccccccccccc}
 & & & \tau_{\ell-1} & & & \cdots & & & \tau_1 & & \tau_0 \;—\; \tau \\
 & & / & & \backslash & / & & \backslash & / & & \backslash\;\; / & \\
\sigma \;—\; \sigma_\ell & & & & \sigma_{\ell-1} & & & & \sigma_2 & & \sigma_1 & \\
 & & & & & & & & & & & \\
\mathbf{a} = \mathbf{b}_\ell & & & & \mathbf{b}_{\ell-1} & & & & \mathbf{b}_2 & & \mathbf{b}_1 & \mathbf{b}_0 = \mathbf{b}
\end{array}$$

in which $\tau_j \in K_i^{\mathbf{b}_j} I$ and $\sigma_j \in K_{i-1}^{\mathbf{b}_j} I$ and

$\tau_0 — \tau$	the simplex τ is boundary-linked to τ_0
$\backslash$	the simplex $\sigma_j \in S_{i-1}^{\mathbf{b}_j}$ for $j = 1, \ldots, \ell - 1$ is chain-linked to τ_j
$/$	the simplex σ_j for $j = 1, \ldots, \ell$ equals $\tau_{j-1} - \lambda_j$
$\sigma — \sigma_\ell$	the simplex $\sigma_\ell \in K_{i-1}^{\mathbf{a}}$ is cycle-linked to σ

Shrubberies, stakes, and hedges

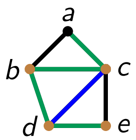
- **shrubbery** $T_i \subseteq K_i$: set of i-faces such that ∂T_i is a $\mathbb{k}$-basis for $\widetilde{B}_{i-1}$
  - shrubbery = spanning tree or spanning forest
- **stake set** $S_{i-1} \subseteq K_{i-1}$: set of $(i-1)$-faces such that $K_{i-1} \setminus S_{i-1}$ gives a basis for $\widetilde{C}_{i-1}/\widetilde{B}_{i-1}$
- **hedge** ST_i: a pair (S_{i-1}, T_i) consisting of a stake set S_{i-1} of dimension $i-1$ and a shrubbery T_i of dimension i



$T_1 = \{ac, bc, bd, de\}$
$S_0 = \{b, c, d, e\}$
$ST_1 = (S_0, T_1)$

Linkages

- Each $\tau \in K_i$ forms a unique T_i**-circuit** $\zeta_{T_i}(\tau) = \tau - t \in \widetilde{Z}_i$ with $t \in \Bbbk\{T_i\}$.
  - τ is **cycle-linked** to every i-face appearing in $\tau - t$
- Each stake $\sigma \in S_{i-1}$ has a unique **shrub** $s \in \Bbbk\{T_i\}$ with ∂s having coefficient 1 on σ and 0 on $S_{i-1} \setminus \sigma$.
  - σ is **chain-linked** to every τ appearing in s
- Each $\rho \in K_i$ has a unique **hedge rim** $r \in \Bbbk\{\bar{S}_i\}$ such that $\rho - r \in \widetilde{B}_i$.
  - ρ is **boundary-linked** to every ρ' appearing in r



$$\zeta_{T_i}(cd) = cd + bc - bd$$
$$s(b) = bc - ac$$
$$r(b) = a$$

Example: $I = \langle xy, y^3, z \rangle$

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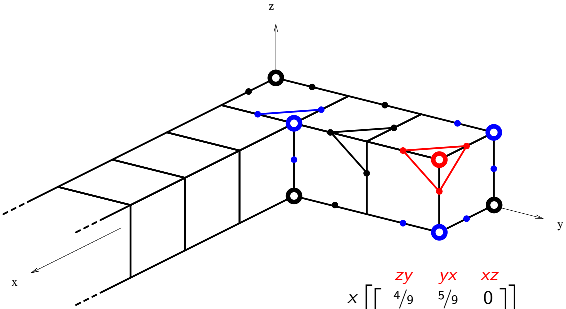
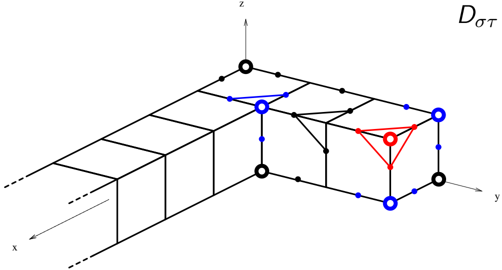


$$\begin{array}{c} \widetilde{H}_{-1}K^{110} \otimes \langle xy \rangle \\ \oplus \\ \widetilde{H}_{-1}K^{030} \otimes \langle y^3 \rangle \\ \oplus \\ \widetilde{H}_{-1}K^{001} \otimes \langle z \rangle \end{array}
\xleftarrow{\begin{array}{c} \quad x\ y\ z \quad\ x\ y \quad\ y\ z \\ \begin{array}{c}\varnothing\\ \varnothing\\ \varnothing\end{array} \begin{bmatrix} [0\ 0\ 1] & [0\ 1] & [0\ 0] \\ [0\ 0\ 0] & [1\ 0] & [0\ 1] \\ [1\ 1\ 0] & [0\ 0] & [1\ 0] \end{bmatrix} \end{array}}
\begin{array}{c} \widetilde{H}_{0}K^{111} \otimes \langle xyz \rangle \\ \oplus \\ \widetilde{H}_{0}K^{130} \otimes \langle xy^3 \rangle \\ \oplus \\ \widetilde{H}_{0}K^{031} \otimes \langle y^3 z \rangle \end{array}
\xleftarrow{\begin{array}{c} \quad zy \quad yx \quad xz \\ \begin{array}{c} x\\ y\\ z\\ x\\ y\\ y\\ z \end{array} \begin{bmatrix} \begin{bmatrix} 4/9 & 5/9 & 0 \\ 1/9 & -1/9 & 0 \\ -5/9 & -4/9 & 0 \end{bmatrix} \\ \begin{bmatrix} -1/2 & 0 & -1/2 \\ 1/2 & 0 & 1/2 \end{bmatrix} \\ \begin{bmatrix} 0 & -1/2 & -1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix} \end{bmatrix} \end{array}}
\widetilde{H}_{1}K^{131} \otimes \langle xy^3 z \rangle$$

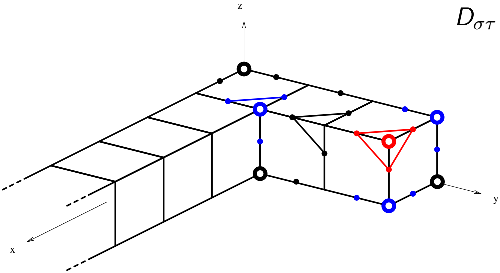
Example: $I = \langle xy, y^3, z \rangle$



$$D_{\sigma\tau} = \sum_{\lambda \in \Lambda(\mathbf{a},\mathbf{b})} \frac{1}{\Delta_{i,\lambda} I} \sum_{\varphi \in \Phi_{\sigma\tau}(\lambda)} w_\varphi$$

λ :	111 ————	121 ————	131
ST_0^λ :	$T_0 = \{x\}$	$T_1 = \{zy, yx\}$	$S_1 = \varnothing$
	or $\{y\}$	$S_0 = \{y, z\}$	
	or $\{z\}$	or $\{x, z\}$	
		or $\{x, y\}$	

Example: $I = \langle xy, y^3, z \rangle$



$$D_{\sigma\tau} = \sum_{\lambda \in \Lambda(\mathbf{a},\mathbf{b})} \frac{1}{\Delta_{i,\lambda} I} \sum_{\varphi \in \Phi_{\sigma\tau}(\lambda)} w_\varphi$$

$yx \quad {\color{red} yx \xrightarrow{1} yx}$

$T_0 = \{y\}$: $x \searrow^{1}$, $y \xrightarrow{-1}$ x $\;/_1 \;\; \backslash^1 \;/_1\;$ x

$T_0 = \{z\}$: $x \nearrow^{1}$, $z \nearrow_{-1}$ $\quad S_0 = \{x, z\}$

$yx \quad {\color{red} yx \xrightarrow{1} yx}$

$T_0 = \{y\}$: $x \searrow^{1}$, $y \xrightarrow{-1}$ x $\;/_1 \;\; \backslash^1 \;/_1\;$ x

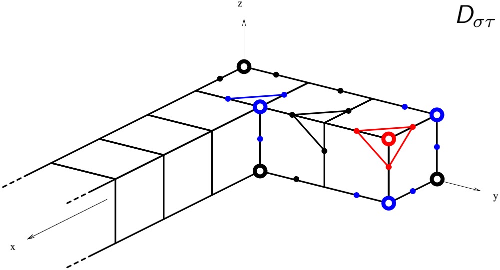
$T_0 = \{z\}$: $x \nearrow^{1}$, $z \nearrow_{-1}$ $\quad S_0 = \{x, y\}$

$zy \quad {\color{red} yx \xrightarrow{1} yx}$

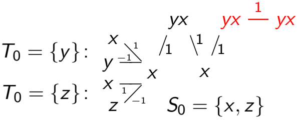
$T_0 = \{x\}$: $z \searrow^{1}$, $x \xrightarrow{-1}$ z $\;/_{-1} \;\; \backslash^1 \;/_1\;$ x

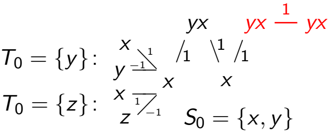
$T_0 = \{y\}$: $z \nearrow^{1}$, $y \nearrow_{-1}$ $\quad S_0 = \{x, y\}$

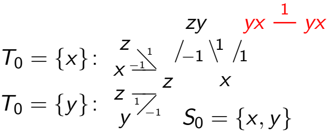
Example: $I = \langle xy, y^3, z \rangle$



$$D_{\sigma\tau} = \sum_{\lambda \in \Lambda(\mathbf{a},\mathbf{b})} \frac{1}{\Delta_{i,\lambda} I} \sum_{\varphi \in \Phi_{\sigma\tau}(\lambda)} w_\varphi$$







$$D_{x,yx} = \tfrac{1}{9}(2 + 2 + 1) = \tfrac{5}{9}$$

Construction of the resolution

- Split the vertical differential in a bicomplex whose vertical, horizontal, and total differentials are Koszul.
- Take the associated Wall complex.
- To get the canonical sylvan resolution, use the Moore-Penrose pseudoinverse.
- Any other splitting induced by a splitting of the boundary map of a simplicial complex yields another minimal free resolution of I.

Splittings from shrubberies and stake sets

- A **splitting** of a complex $C_\bullet$ consists of a differential

$$d^+ = d_i^+ : C_i \to C_{i+1}$$

  such that $dd^+d = d$ and $d^+dd^+ = d^+$.

- This is equivalent to a direct sum decomposition $C_i = B'_{i-1} \oplus H_i \oplus B_i$, where B_i is the image $d(C_{i+1})$, H_i is isomorphic to $H_i(C_\bullet)$, and B'_{i-1} is isomorphic to B_{i-1}.
- Each hedge $ST_i = (S_{i-1}, T_i)$ defines a **hedge splitting** $d^+_{ST_i} : C_{i-1} \to C_i$ via
  1. $d^+d(t) = t$ for all $t \in T_i$
  2. $d^+(s) = 0$ for all $s \in \bar{S}_{i-1}$
- A **community** is a sequence of hedges $ST_\bullet = (ST_0, ST_1, ST_2, \ldots)$ such that $T_i \cap S_i = \emptyset$, and it defines a differential d^+ comprised of hedge splittings.

Hedge splittings

- Each hedge $ST_i = (S_{i-1}, T_i)$ defines a **hedge splitting** $d^+_{ST_i} : C_{i-1} \to C_i$ via
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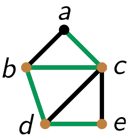


$T_1 = \{ac, bc, bd, de\}$
$S_0 = \{b, c, d, e\}$
$ST_1 = (S_0, T_1)$

- $d^+(b) = d^+(b - a) = d^+(d(ac - bc)) = ac - bc$

Minimal free resolutions from hedge splittings

Theorem (Eagon-Miller-O. 2019)

Fix a monomial ideal I. Any hedge splittings $d_{\mathbf{b}}^{+}$ of the boundary maps $d_{\mathbf{b}}$ of the Koszul simplicial complexes $K^{\mathbf{b}}I$ yield a minimal free resolution of I whose differential from homological stage $i+1$ to stage i has its component $\widetilde{H}_i K^{\mathbf{b}} I \otimes \Bbbk[\mathbf{x}](-\mathbf{b}) \to \widetilde{H}_{i-1} K^{\mathbf{a}} I \otimes \Bbbk[\mathbf{x}](-\mathbf{a})$ induced by the map

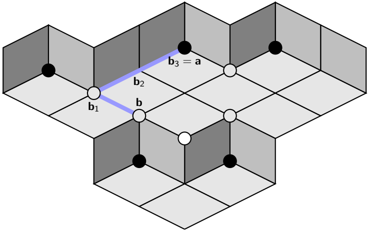
$$D : \widetilde{H}_i K^{\mathbf{b}} I \to \widetilde{H}_{i-1} K^{\mathbf{a}} I$$

in $\mathbb{N}^n$-degree $\mathbf{b}$ that acts on any i-cycle in $\widetilde{Z}_i K^{\mathbf{b}} I$ via

$$D = \sum_{\lambda \in \Lambda(\mathbf{a},\mathbf{b})} (I^{\mathbf{a}} - d_i^{\mathbf{a}+} d_i^{\mathbf{a}}) d_1^{\lambda_\ell} \Big(\prod_{j=1}^{\ell-1} d_i^{\mathbf{b}_j +} d_1^{\lambda_j} \Big) (I^{\mathbf{b}} - d_{i+1}^{\mathbf{b}} d_{i+1}^{\mathbf{b}+}),$$

where $d_1 = d_1^{e_1} + d_1^{e_2} + \ldots + d_1^{e_n}$ acts as the boundary operator, and $\lambda_j = e_k$ for some k.

Lattice paths



$$\widetilde{C}_{i-1}K^{\mathbf{a}}I \xleftarrow{d_1^{\lambda_\ell}}$$

$$\partial_i^{\mathbf{a}+} \uparrow \downarrow \partial_i^{\mathbf{a}} \qquad \uparrow \partial_i^{\mathbf{b}_{\ell-1}+}$$

$$\widetilde{C}_{i-1}K^{\mathbf{b}_{\ell-1}}I \xleftarrow{d_1^{\lambda_{\ell-1}}}$$

$$\ddots \qquad \uparrow \partial_i^{\mathbf{b}_2+}$$

$$\widetilde{C}_{i-1}K^{\mathbf{b}_2}I \xleftarrow{d_1^{\lambda_2}}$$

$$\uparrow \partial_i^{\mathbf{b}_1+} \qquad \partial_{i+1}^{\mathbf{b}} \downarrow \uparrow \partial_{i+1}^{\mathbf{b}+}$$

$$\widetilde{C}_{i-1}K^{\mathbf{b}_1}I \xleftarrow{d_1^{\lambda_1}} \widetilde{C}_iK^{\mathbf{b}}I$$

$$D = \sum_{\lambda \in \Lambda(\mathbf{a},\mathbf{b})} (I^{\mathbf{a}} - \partial_i^{\mathbf{a}+}\partial_i^{\mathbf{a}})d_1^{\lambda_\ell}\left(\prod_{j=1}^{\ell-1} \partial_i^{\mathbf{b}_j+} d_1^{\lambda_j}\right)(I^{\mathbf{b}} - \partial_{i+1}^{\mathbf{b}}\partial_{i+1}^{\mathbf{b}+}),$$

$$d_1 = d_1^{e_1} + d_2^{e_2} + \cdots + d_1^{e_n}$$

Resolutions of stable ideals

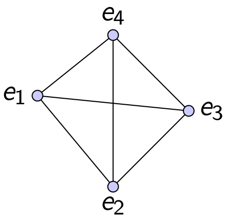
- For a monomial $\mathbf{x}^{\mathbf{b}}$, let $m(\mathbf{b})$ be the maximum index of a nonzero entry of $\mathbf{b}$.
- A monomial ideal I is **stable** if for every monomial $\mathbf{x}^{\mathbf{b}} \in I$, $\mathbf{x}^{\mathbf{b}-e_{m(\mathbf{b})}+e_i} \in I$ for all $1 \leq i < m(\mathbf{b})$.
  - Example: $I = \langle x_1^2, x_1x_2, x_2^2, x_2x_3 \rangle$
- An **admissible symbol** is of the form $e(\sigma, u)$, where u is a generator of I and $m(\sigma) < m(u)$.
- The **Eliahou–Kervaire resolution** [Eliahou-Kervaire 1990] for stable ideals has free modules F_q, which are $\mathbb{k}[\mathbf{x}]$-modules generated by the admissible symbols $e(\sigma, u)$ with $|\sigma| = q$.
- The differential $d : F_q \to F_{q-1}$ is given by

$$d(e(\sigma, u)) = \sum_{r=1}^{q} x_{i_r}(-1)^r e(\sigma_r, u) - \sum_{r \in A(\sigma;u)} (-1)^r y_r e(\sigma_r, u_r).$$

- Recall: $K^{\mathbf{b}}I = \{\text{squarefree } \tau \mid \mathbf{x}^{\mathbf{b}-\tau} \in I\}$
  - If $e_{m(\mathbf{b})} \not\leq \tau$, then $m(\mathbf{b}) = m(\tau)$ and $\mathbf{x}^{\mathbf{b}-\tau-e_{m(\mathbf{b})}+e_i} \in I$, so $\tau + e_{m(\mathbf{b})} - e_i \in K^{\mathbf{b}}I$
- If I is stable, $K^{\mathbf{b}}I$ is a **near-cone**.

Near-cones

- A simplicial complex Δ on the vertices $\{e_1, \ldots, e_n\}$ is a *near-cone* if for every $\tau \in \Delta$ such that $e_n \not\preceq \tau$, then $\tau - e_j + e_n \in \Delta$ for all $e_j \preceq \tau$.
- Example:



Hedges in near-cones

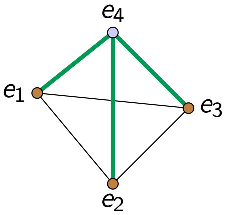
- Proposition (with Eagon and Miller): In the Koszul complex $K^{\mathbf{b}}I$, the set

$$S_{i-1} = \{\tau \mid e_{m(\mathbf{b})} \not\leq \tau, \tau + e_{m(\mathbf{b})} \in K^{\mathbf{b}}I\}$$

  for faces τ of dimension $i-1$ is a stake set of dimension $i-1$, and the set

$$T_i = \{\tau + e_{m(\mathbf{b})} \mid \tau \in S_{i-1}\}$$

  is a shrubbery of dimension i.
- Example: $S_0 = \{e_1, e_2, e_3\}$, $T_1 = \{e_1e_4, e_2e_4, e_3e_4\}$

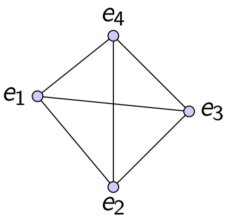


Splittings in near-cones

- Proposition (with Eagon and Miller): Let $ST_i = (S_{i-1}, T_i)$, where $S_{i-1} = \{\tau \mid e_{m(\mathbf{b})} \not\preceq \tau, \tau + e_{m(\mathbf{b})} \in K^{\mathbf{b}}I\}$ and $T_i = \{\tau + e_{m(\mathbf{b})} \mid \tau \in S_{i-1}\}$. Then

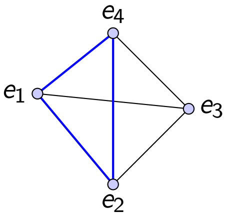
$$d^+_{ST_i}(\tau) = \begin{cases} (-1)^{|\tau|}(\tau + e_{m(\mathbf{b})}) & \text{if } m(\tau) < m(\mathbf{b}) \\ & \text{and } \tau + e_{m(\mathbf{b})} \in K^{\mathbf{b}}I \\ 0 & \text{otherwise.} \end{cases}$$

- Example: $d^+_{ST_1}(e_1) = -e_1e_4, d^+_{ST_1}(e_4) = 0$

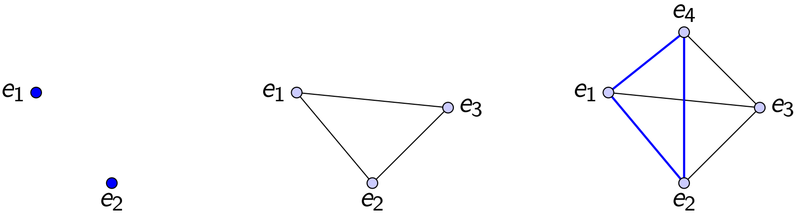


A canonical basis for homology

- The hedge splitting $d^+_{ST_i}$ gives a canonical basis for $\widetilde{H}_{i-1}(K^{\mathbf{b}}I; \Bbbk)$ (computed via the projection $I - dd^+ - d^+d$).
- For each admissible symbol $e(\sigma, u)$ such that $\sigma + u = \mathbf{b}$, the basis element is $(-1)^{|\sigma|}\sigma + \sum_j c_j\tau_j$, the boundary of the face $(\sigma + e_{m(\mathbf{b})})$.
- Example: e_1e_2 corresponds to the admissible symbol $e(e_1e_2, e_3e_4)$ and the canonical basis element $e_1e_2 - e_1e_4 + e_2e_4$



The sylvan resolution



$K^{\mathbf{b}-e_4-e_3} I$ $\quad$ $K^{\mathbf{b}-e_4} I$ $\quad$ $K^{\mathbf{b}} I$

$$e_1 - e_2 \xleftarrow{d_1^{e_3}} -e_1 e_3 + e_2 e_3$$

$$\uparrow d^+$$

$$e_1 - e_2 \xleftarrow{d_1^{e_4}} e_1 e_2 - e_1 e_4 + e_2 e_4$$

Concordance with the Eliahou–Kervaire resolution

- When computing the sylvan differential, the only lattice paths that give nonzero coefficients in the image are all lattice paths of length one that move back in the direction of faces of σ and lattice paths $(\mathbf{b}_\ell, \ldots, \mathbf{b}_1, \mathbf{b}_0 = \mathbf{b})$ where $\mathbf{b}_{i-1} - \mathbf{b}_i = m(\mathbf{b}_{i-1})$.
- Theorem (with Eagon and Miller): The sylvan resolution induced by the hedge splittings given earlier is the Eliahou–Kervaire resolution.

Thank you!

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