Bi-Lipschitz embeddings of Grushin spaces

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The bi-Lipschitz embedding problem

**Definition**

A map \( f : (X, d_X) \to (Y, d_Y) \) is bi-Lipschitz if

\[
C^{-1}d_X(x, y) \leq d_Y(f(x), f(y)) \leq Cd_X(x, y)
\]

for some \( C > 1 \) and all \( x, y \in X \).

**Question**

What conditions on a metric space imply that it can or cannot be embedded in a well-known model space (e.g. Euclidean spaces, hyperbolic spaces) under a bi-Lipschitz mapping?
Historical summary

- Seminal result by Assouad [Ass83] (discussed below)
- Studied intensively in series of papers by Semmes in 1990’s (e.g. [Sem93] [Sem96] [Sem99]) using ideas from harmonic analysis and fractal geometry; ‘easy conjectures’ shown false
- Studied by Lang and Plaut [LP01] using ideas from Alexandrov geometry
- Lafforgue and Naor [LN14], others
Assouad’s Theorem

**Definition**

A metric space $X$ is *doubling* if there is a $C > 1$ such that every ball $B(x, r)$ can be covered by $C$ balls of radius $r/2$.

**Theorem ([Ass83])**

Let $(X, d)$ be a doubling metric space. Then, for all $\alpha \in (0, 1)$, the “snowflaked” space $(X, d^\alpha)$ can be bi-Lipschitz embedded in a finite-dimensional Euclidean space.

- Snowflaking distorts geometry of $X$: rectifiability of curves, Hausdorff dimension.
The Grushin plane

- Given $\alpha \geq 0$, the vector fields $X = \partial_x$ and $Y = |x|^{\alpha} \partial_y$ define a sub-Riemannian metric on $\mathbb{R}^2$.
- $X|_{(x,y)}$, $Y|_{(x,y)}$ form an orthonormal basis for the tangent space at each point in $\mathbb{R}^2 \setminus \{(x, y) : x = 0\}$.
- There are higher-dimensional Grushin spaces; these are defined similarly but with more vector fields.

(a) Vector fields $X$, $Y$  
(b) Geodesics from origin
The Grushin plane

- Length of path $\gamma$ given by
  \[
  \ell(\gamma) = \int_{\gamma} \sqrt{dx^2 + |x|^{-2\alpha} dx^2}
  \]

- Distance between two points $z_1, z_2$ is $d_\alpha(z_1, z_2) = \inf \ell(\gamma)$ taken over all absolutely continuous paths from $z_1$ to $z_2$.

- Resulting metric space $(\mathbb{G}_\alpha^2, d_\alpha)$ is Riemannian manifold except on singular line $\{(x, y) : x = 0\}$; topologically it is $\mathbb{R}^2$. 
Results on Grushin plane

- Observed by Semmes that Heisenberg group cannot be embedded in Euclidean space, even locally (cf. Cheeger’s differentiability theorem) [Sem96]
- Grushin plane studied by Seo [Seo11] in her thesis; shown to be embeddable
- Grushin plane shown by Meyerson [Mey11] to be quasisymmetrically equivalent to Euclidean plane
- Explicit embedding yielding optimal target dimension of 3 constructed by Wu [Wu15]; generalized to $\alpha$-Grusin plane by R. and Vellis [RVa]
Seo’s embeddability criterion

**Theorem ([Seo11])**

A doubling metric space \((X, d)\) admits a bi-Lipschitz embedding into some Euclidean space if and only if the following hold:

1. There is a closed subset \(Y\) of \(X\) which admits a bi-Lipschitz embedding into some \(\mathbb{R}^{M_1}\).
2. There is a Christ-Whitney decomposition of \(\Omega = X \setminus Y\) such that each cube admits a bi-Lipschitz embedding into some \(\mathbb{R}^{M_2}\) with uniform bi-Lipschitz constant.

- Used by Seo to prove embeddability of the Grushin plane
Let $X$ be a metric space. For an open subset $\Omega \subsetneq X$, a
**Christ-Whitney decomposition of $\Omega$ with data** $(\delta, c_0, C_1, a)$, where
$0 < \delta < 1$, $0 < c_0 < C_1$, $a \geq 4$, is a collection $M_\Omega$ of disjoint open
subsets of $X$ satisfying the following properties:

1. $\bigcup M_\Omega$ is dense in $\Omega$.
2. For any $Q \in M_\Omega$, there exists $x \in \Omega$ and $k \in \mathbb{Z}$ such that
   $B(x, c_0\delta^k) \subset Q \subset B(x, C_1\delta^k)$.
3. $(a - 2)C_1\delta^k \leq \text{dist}(Y, Q) \leq \left(\frac{aC_1}{\delta}\right)\delta^k$.

It is straightforward to verify any proper open subset of a
doubling metric space has a Christ-Whitney decomposition,
subject to mild restrictions on the data.
Define the map \( \varphi : \mathbb{G}_\alpha^2 \rightarrow \mathbb{R}^2 \) by
\[
(u, v) = \varphi(x, y) = \left(\frac{1}{1 + \alpha} |x|^{\alpha} x, y \right).
\]

This map appears in [Bec01], [MM04], [Mey11].

The push-forward of the \( \alpha \)-Grushin line element under \( \varphi \) is
\[
ds' = \frac{1}{(1 + \alpha)^{\alpha/(1+\alpha)} |u|^{\alpha/(1+\alpha)}} \sqrt{du^2 + dv^2}.
\]
Definition of conformal Grushin spaces

Definition

Let \( n \in \mathbb{N} \), let \( Y \subset \mathbb{R}^n \) be a nonempty closed set, and let \( \beta \in [0, 1) \). The \((Y, \beta)\)-Grushin space is the space \( \mathbb{R}^n \) equipped with the metric determined by the line element

\[
ds = \frac{ds_E}{d_E(\cdot, Y)^\beta}.
\]

The \((Y, \beta)\)-Grushin metric is denoted here by \( d_Y \). The Euclidean metric is denoted by \( d_E \).  

- Similar metric on proper domain \( \Omega \subset \mathbb{R}^2 \) considered by Gehring–Martio [GM85], Lappalainen [Lap85], Langmeyer [Lan98] (sub-quasihyperbolic metric)
- Take \( Y = \{0\} \subset \mathbb{R}^2 \) as singular set. The \((Y, \beta)\)-Grushin plane is path-isometric to a cone in \( \mathbb{R}^3 \) with angular defect \( 2\pi \beta \).
Theorem

Let $n \in \mathbb{N}$, $Y \subset \mathbb{R}^n$ be closed, nonempty, and $\beta \in [0, 1)$. If the $(Y, \beta)$-Grushin space satisfies the Hölder condition $d_Y(x, y) \leq H d_E(x, y)^{1-\beta}$ for some $H > 0$ and all $x, y \in \mathbb{R}^n$, then the $(Y, \beta)$-Grushin space admits a bi-Lipschitz embedding in some Euclidean space of sufficiently high dimension.

- In the following, we will always assume that the $(Y, \beta)$-Grushin space satisfies this Hölder condition.
Proposition

Let $X \subset \mathbb{R}^n$ be a nonempty closed set such that $\Omega = \mathbb{R}^n \setminus X$ is the union of finitely many uniform domains and $\overline{\Omega} = \mathbb{R}^n$. Then for all $\beta \in [0,1)$ and any nonempty closed subset $Y \subset X$, the $(Y, \beta)$-Grushin space satisfies the Hölder condition of the main theorem.
Proof is an application of Seo’s embedding criterion. We must check:
(1) doubling property
(2) embeddability of singular set $Y$
(3) uniform embedding of Christ-Whitney cubes

(2) is immediate by Assouad’s theorem since metric on singular set is bi-Lipschitz equivalent to snowflake of Euclidean metric.

(3) is technical though elementary: we show that $d_Y \simeq d_E(Q, Y)^{-\beta} d_E$ for each cube $Q$ of sufficiently fine Christ-Whitney decomposition.
Quasisymmetric parametrization

1. Follows from proving a stronger result that the identity map is a quasisymmetry from $\mathbb{R}^n$ to the $(Y, \beta)$-Grushin space.

2. A topological embedding $f : (X, d_X) \to (Y, d_Y)$ is quasisymmetric if there exists a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that

$$\frac{d_Y(f(x), f(y))}{d_Y(f(x), f(z))} \leq \eta(t)$$

whenever the distinct points $x, y, z \in X$ satisfy $d_X(x, y) \leq td_X(x, z)$.

3. Quasisymmetric maps preserve the doubling property of a space.
References


