NON-EMBEDDABILITY OF THE URYSOHN SPACE INTO SUPERREFLEXIVE BANACH SPACES

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Abstract. We present Pestov's proof that the Urysohn space does not embed uniformly into a superreflexive Banach space ([P]). Its interest lies mainly in the fact that the argument is essentially combinatorial. Pestov uses the extension property for the class of finite metric spaces ([S2]) to build affine representations of the isometry group of the Urysohn space.

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1. Uniform embeddings

We recall the notion of uniform embedding of metric spaces.

Let X and Y be two metric spaces. A **uniform embedding** of X into Y is an embedding of X into Y as uniform spaces. Equivalently, a map $f : X \to Y$ is a uniform embedding if there exist two non-decreasing functions ρ_1 and ρ_2 from $\mathbb{R}_+$ to $\mathbb{R}_+$, with $0 < \rho_1 \leqslant \rho_2$ and $\lim_{r\to 0} \rho_2(r) = 0$, such that for all x, x' in X, one has

$$\rho_1(d_X(x, x')) \leqslant d_Y(f(x), f(x')) \leqslant \rho_2(d_X(x, x')).$$

In particular, a uniform embedding is uniformly continuous.

Uniform embeddings transpose the local structure of metric spaces: what matter are small neighborhoods of points. We are interested in the existence of uniform embeddings into nice Banach spaces, where niceness begins at reflexivity.

2. The Urysohn space

The Urysohn space $\mathbb{U}$ is a **universal** Polish space: it is a complete separable metric space that contains an isometric copy of every (complete) separable metric space. Moreover, the Urysohn space is remarkable for its strong homogeneity properties: up to isometry, it is the unique Polish space that is both universal and ultrahomogeneous.

Definition 2.1. A metric space X is **ultrahomogeneous** if every isometry between finite subsets of X extends to a global isometry of X.

The space $\mathbb{U}$ was built by Urysohn in the early twenties ([U1]), but was long forgotten after that. Indeed, another universal Polish space, $\mathcal{C}([0,1],\mathbb{R})$ (Banach-Mazur, see [B] and [S1]), put the Urysohn space in the shade for sixty years. It regained interest in the eighties when Katětov ([K2]) provided a new construction of the Urysohn space. From this construction, Uspenskij ([U2]) proved that not only is $\mathbb{U}$ universal but also its isometry group[1] is a universal Polish group (every Polish group embeds in $\mathrm{Iso}(\mathbb{U})$ as a topological subgroup).

We will see that in fact, the Urysohn space enjoys a much stronger homogeneity property than ultrahomogeneity. In the next section, we will present this strengthening of ultrahomogeneity.

First, let us present Katětov's construction of the Urysohn space and explain how it yields the universality of its isometry group.

2.1. **Katětov spaces.** Let X be a metric space.

Definition 2.2. A **Katětov map** on X is a map $f : X \to \mathbb{R}^+$ such that for all x and x' in X, one has

$$|f(x) - f(x')| \leqslant d(x,x') \leqslant f(x) + f(x').$$

A Katětov map corresponds to a metric one-point extension of X: if f is a Katětov map on X, then we can define a metric on $X \cup \{f\}$ that extends the metric on X by putting, for all x in X,

$$d(f,x) = f(x).$$

This will indeed be a metric because Katětov maps are exactly those which satisfy the triangle inequality.

Example 2.3. If x is a point in X, then the map $\delta_x : X \to \mathbb{R}^+$ defined by $\delta_x(x') = d(x,x')$ is a Katětov map on X. It correspond to a trivial extension of X: we are adding the point x to X.

We denote by $E(X)$ the space of all Katětov maps on X. We equip the space $E(X)$ with the supremum metric, which geometrically represents the smallest possible distance between the two extension points.

The maps δ_x of example 2.3 define an isometric embedding of the space X into $E(X)$. We therefore identify X with its image in $E(X)$ via this embedding. This observation will allow us to build towers of extensions in the next section. The essential property of those towers is the following.

Proposition 2.4. Every isometry of X extends uniquely to an isometry of $E(X)$.

In particular, the uniqueness implies that the extension defines a group homomorphism from $\mathrm{Iso}(X)$ to $\mathrm{Iso}(E(X))$.

[1] Isometry groups are endowed with the topology of pointwise convergence. Basic open sets are the sets of all isometries that extend a given partial isometry between finite subsets. When X is a complete separable metric space, its isometry group $\mathrm{Iso}(X)$ is a Polish group.

Proof. Let φ be an isometry of X. If ψ extends φ, we must have $d(\psi(f), \delta_x) = d(f, \delta_{\varphi^{-1}(x)}) = f(\varphi^{-1}(x))$ for all x in X and f in $E(X)$, hence the uniqueness.

Thus, we extend φ to the space $E(X)$ by putting $\psi(f) = f \circ \psi^{-1}$ for all f in $E(X)$. It is easy to check that the map ψ is an isometry of $E(X)$ that extends φ. □

In general, the space $E(X)$ is unfortunately not separable. Since we are interested only in Polish spaces, we circumvent this problem by considering only Katětov maps with finite support.

Definition 2.5. Let S be a subset of X and let f be a Katětov map on X. We say that S is a **support** for f if for all x in X, we have

$$f(x) = \inf_{y \in S} f(y) + d(x, y).$$

In other words, S is a support for f if the map f is the biggest 1-Lipschitz map on X that coincides with f on S.

We denote by $E(X, \omega)$ the space of all Katětov maps that admit a finite support[2]. If the metric space X is separable, then $E(X, \omega)$ is separable, it still embeds X isometrically, and isometries of X still extend uniquely to isometries of $E(X, \omega)$. Moreover, the extension homomorphism from $\mathrm{Iso}(X)$ to $\mathrm{Iso}(E(X, \omega))$ is continuous (see [M2, proposition 2.5]).

2.2. **Tower construction of the Urysohn space.** The construction of the Urysohn space we present highlights its universality: we start with an arbitrary Polish space and we build a copy of the Urysohn space around it. Besides, the construction keeps track of the isometries of the original Polish space, which points to the universality of its isometry group as well.

Let X be our starting Polish space. We build an increasing sequence (X_n) of metric spaces recursively, by setting

- $X_0 = X$;
- $X_{n+1} = E(X_n, \omega)$.

The discussion above guarantees that isometries extend continuously at each step: every isometry of X_n extends to an isometry of X_{n+1} and the extension homomorphism from $\mathrm{Iso}(X_n)$ to $\mathrm{Iso}(X_{n+1})$. Thus, if we write $X_\infty = \bigcup_{n \in \mathbb{N}} X_n$, we obtain a continuous extension homomorphism from $\mathrm{Iso}(X)$ to $\mathrm{Iso}(X_\infty)$.

Now, consider the completion $\widehat{X_\infty}$ of X_∞. Since all the X_n are separable, the space $\widehat{X_\infty}$ is Polish. Moreover, isometries of X_∞ extend to isometries of $\widehat{X_\infty}$ by uniform continuity, so we get a continuous extension homomorphism from $\mathrm{Iso}(X)$ to $\mathrm{Iso}(\widehat{X_\infty})$.

It remains to explain why the space $\widehat{X_\infty}$ is the promised ultrahomogeneous and unique Urysohn space. The key defining property of $\widehat{X_\infty}$ is that every one-point metric extension of a finite subset of $\widehat{X_\infty}$ is realized in $\widehat{X_\infty}$ over this finite set.

Definition 2.6. A metric space X is said to have the **Urysohn property** if for every finite subset A of X and every Katětov map $f \in E(A)$, there exists x in X such that for all a in A, we have $d(x, a) = f(a)$.

Theorem 2.7. (Urysohn) Let X be a complete separable metric space. If X has the Urysohn property, then X is ultrahomogeneous.

Proof. We carry a back-and-forth argument. Let $i : A \to B$ an isometry between two finite subsets of X. Enumerate a dense subset $\{x_n : n \geqslant 1\}$ of X. Recursively, we build finite subsets A_n and B_n of X and isometries $i_n : A_n \to B_n$ such that

[2]The letter ω is the set-theoretic name for $\mathbb{N}$.

- $A_0 = A$ and $B_0 = B$;
- $i_0 = i$;
- $A_n \subseteq A_{n+1}$ and $B_n \subseteq B_{n+1}$;
- $x_n \in A_n \cap B_n$;
- i_{n+1} extends i_n.

To this aim, assume A_n and B_n have been built. Consider the metric extension of A_n by x_{n+1}: the corresponding Katětov map is $\delta_{x_{n+1}}$. We push it forward to a Katětov map on B_n via the isometry i_n. Now, since the space X satisfies the Urysohn property, we can find an element y_{n+1} that realizes it; we add it to B_n and extend i_n by setting $i'_{n+1}(x_{n+1}) = y_{n+1}$. This constitutes the *forth* step.

For the *back* step, we apply the same argument to the inverse of the isometry i'_{n+1} to find a preimage to x_{n+1}.

In the end, the union of all the isometries i_n defines an isometry of a dense subset of X, so it extends to an isometry of the whole space X (because X is complete). This is the desired extension of i. ☐

Another back-and-forth argument shows that any two complete separable metric spaces with the Urysohn property are isomorphic (see [G, theorem 1.2.5]). Thus, we may for instance define the Urysohn space $\mathbb{U}$ to be the space obtained from $X = \{0\}$ by applying the tower construction above. This uniqueness result guarantees that $\mathbb{U}$ indeed embeds every Polish space isometrically. Moreover, the construction also yields that its isometry group $\mathrm{Iso}(\mathbb{U})$ embeds all isometry groups of Polish spaces. A beautiful result of Gao and Kechris ([GK]) states that these actually encompass all Polish groups, so we conclude that $\mathrm{Iso}(\mathbb{U})$ is a universal Polish group.

In particular, $\mathrm{Iso}(\mathbb{U})$ contains the group $\mathrm{Homeo}_+[0,1]$ of orientation-preserving homeomorphisms of the unit interval. In the proof of theorem 7.1, we will use this fact, together with the following result of Megrelishvili ([M1]), to show that the Urysohn space does not admit any uniform embedding into a superreflexive Banach space.

Theorem 2.8. (Megrelishvili) The only continuous representation of $\mathrm{Homeo}_+[0,1]$ by linear isometries on a reflexive Banach space is the trivial representation.

3. The extension property

In 1992, Hrushovski ([H2]) proved that for every finite graph, there exists a bigger finite graph such that every partial graph isomorphism of the smaller graph extends to a global graph automorphism of the bigger graph. It turns out that this phenomenon occurs in several other structures, and in particular for metric spaces.

Definition 3.1. A metric space has the **extension property** if for every finite subset A of X, there exists a finite subset B of X that contains A such that every partial isometry of A extends to a global isometry of B.

The extension property is indeed a strengthening of ultrahomogeneity.

Proposition 3.2. Let X be a complete separable metric space. If X has the extension property, then X is ultrahomogeneous.

Proof. Let $i : A \to B$ be an isometry between two finite subsets A and B of X. We wish to extend i to a global isometry of X. First, the extension property gives a finite subset Y_0 of X containing A and B such that the partial isometry i extends to a global isometry j_0 of Y_0.

Enumerate a dense subset $\{x_n : n \geqslant 1\}$ of X. Recursively, we build an increasing chain of finite subsets Y_n of X, with $Y_{n+1} \supseteq Y_n \cup \{x_n\}$, and an increasing chain of global isometries j_n of Y_n by applying the extension property.

Let now Y be the union of all the Y_n's. The map j defined by $j(x) = j_n(x)$ if $x \in Y_n$ is a global isometry of Y. Since Y contains all the points x_n, it is dense in X, so j extends to an isometry of the whole space X (because X is complete). □

Independently, Vershik ([V]) announced and Solecki ([S2]) proved that the Urysohn space satisfies the extension property. Consequently, the extension property is sometimes also called the *Hrushovski-Solecki-Vershik property.* Note that this is really a result about the class of *all* metric spaces. It means that for every finite metric space, there exists a bigger finite metric space such that every partial isometry of the smaller metric space extends to a global isometry of the bigger metric space.

In fact, the Urysohn space satisfies an even stronger form of extension property ([S3]): we can choose the extension of those partial isometries to be compatible with the group structure. Thus, the extension will provide a group homomorphism from the isometry group of the smaller metric space to the isometry group of the bigger one. This **coherent extension property** has a very powerful consequence on the isometry group, which is the heart of the argument for theorems 6.1 and 5.5.

Proposition 3.3. Let X be a complete separable metric space. If X satisfies the coherent extension property, then its isometry group $\mathrm{Iso}(X)$ contains a dense locally finite subgroup.

A group is said to be **locally finite** if every finitely generated subgroup is finite.

Proof. We carry the same construction as in the proof of proposition 3.2: we recursively build finite subsets Y_n of X such that

- $Y_n \subseteq Y_{n+1}$;
- every partial isometry of Y_n extends to a global isometry of Y_{n+1};
- (coherence) moreover, the extension defines a group embedding from $\mathrm{Iso}(Y_n)$ to $\mathrm{Iso}(Y_{n+1})$;
- the union $Y = \bigcup_{n\in\mathbb{N}} Y_n$ of all the Y_n's is dense in X.

Since the extension is coherent, the union $G = \bigcup_{n\in\mathbb{N}} \mathrm{Iso}(Y_n)$ is an increasing union of subgroups of $\mathrm{Iso}(Y)$. Thus, as the increasing union of finite groups, it is a locally finite group. We show that the group G is dense in $\mathrm{Iso}(Y)$. By density of Y in X, the group $\mathrm{Iso}(Y)$ is dense in $\mathrm{Iso}(X)$, so this will complete the proof.

Consider a basic open set in $\mathrm{Iso}(Y)$. It is given by a partial isometry $i : A \to B$ between finite subsets of Y. Since A and B are finite, there exists an integer n such that both A and B are contained in Y_n. But then the partial isometry i of Y_n extends to a global isometry of Y_{n+1}, which is in G. Thus, the basic open set contains an element of G, and G is indeed dense in $\mathrm{Iso}(Y)$. □

Remark 3.4. In [P], Pestov states the above result for metric spaces which satisfy only the extension property, without any coherence assumption. It is not clear, then, how to build a dense locally finite subgroup recursively, as the groups $\mathrm{Iso}(Y_n)$ need not even be subgroups of $\mathrm{Iso}(Y)$, nor be included in one another.

4. Ultrapowers of Banach spaces

4.1. **Ultrafilters.** Dually to ideals giving a notion of smallness, ultrafilters give a way to declare some sets as *large*. More precisely, a **filter** on a set I is a collection $\mathcal{F}$ of subsets of I such that

- (non-triviality) the whole set I is in $\mathcal{F}$ but the empty set is not in $\mathcal{F}$;
- if A is in $\mathcal{F}$, then any subset B of I containing A also is in $\mathcal{F}$;
- the intersection of two elements of $\mathcal{F}$ is again in $\mathcal{F}$.

An **ultrafilter** is a maximal filter (with respect to inclusion). Equivalently, a filter $\mathcal{U}$ on I is an ultrafilter if and only if for each subset A of I, either A is in $\mathcal{U}$ or $I \setminus A$ is in $\mathcal{U}$.

The point of ultrafilters, aside from brewing ultracoffee, is to make arbitrary sequences converge.

Definition 4.1. Let X be a topological space. Let I be a set (of indices) and let $\mathcal{F}$ be a filter on I. Let $(x_i)_{i\in I}$ be a family of elements of X and let x be a point in X. We say that x is the **limit** of $(x_i)_{i\in I}$ **along** $\mathcal{F}$, and we write $x = \lim\limits_{i\to\mathcal{F}} x_i$, if for every neighborhood V of x in X, the set $\{i \in I : x_i \in V\}$ is in $\mathcal{F}$.

The usual notion of convergence for sequences indexed by the integers thus corresponds to the convergence along the filter of cofinite subsets of $\mathbb{N}$: this filter contains all the intervals $[n;\infty[$.

Proposition 4.2. Let $(x_i)_{i\in I}$ be a family of elements of reals and let $\mathcal{U}$ be an ultrafilter on I. If $(x_i)_{i\in I}$ is bounded, then the family $(x_i)_{i\in I}$ has a limit along $\mathcal{U}$.

Proof. We use the classical Bolzano-Weierstrass cutting-in-half argument. Assume that the family takes its values in the bounded interval $[a,b]$. Cut the interval in two and look at which elements of the sequence fall in which half: consider the two sets

$$L = \left\{ i \in I : x_i \in \left[a, \frac{a+b}{2} \right] \right\} \text{ and } R = \left\{ i \in I : x_i \in \left] \frac{a+b}{2}, b \right] \right\}.$$

Since $\mathcal{U}$ is an ultrafilter, exactly one of the sets L and R belongs to $\mathcal{U}$, say L.

Then we do that again in L: we consider the sets

$$L' = \left\{ i \in I : x_i \in \left[a, \frac{3a+b}{4} \right] \right\} \text{ and } R' = \left\{ i \in I : x_i \in \left] \frac{3a+b}{4}, \frac{a+b}{2} \right] \right\}.$$

This time, either L' is in $\mathcal{U}$, or its complement, which is $R' \cup R$ is. But we know that L is in the ultrafilter $\mathcal{U}$ too, so the intersection $L \cap (R' \cup R) = R'$ belongs to $\mathcal{U}$; and so on.

Thus, inductively, we find a decreasing sequence of intervals $[a_n, b_n]$ of length $\dfrac{b-a}{n}$ such that for all n, the set $\{i \in I : x_i \in [a_n, b_n]\}$ is in the ultrafilter $\mathcal{U}$. It follows that the intersection point of all those intervals $[a_n, b_n]$ is the limit of the family $(x_i)_{i\in I}$ along the ultrafilter $\mathcal{U}$. □

The same argument readily adapts to families in any compact space (see e.g. [E2, theorem 3.1.24]).

4.2. **Ultraproducts of metric spaces.** Let $(X_i)_{i\in I}$ be a family of metric spaces. We choose a distinguished point x_i in each X_i. We consider the following subset of the product of the X_i's:

$$\ell^\infty(X_i, x_i, I) = \{y \in \prod_{i\in I} X_i : \sup_{i\in I} d_{X_i}(x_i, y_i) < \infty\}.$$

Let $\mathcal{U}$ be an ultrafilter on I. The boundedness assumption above allows us to equip $\ell^\infty(X_i, x_i, I)$ with the following pseudometric:

$$d(y,z) = \lim_{i\to\mathcal{U}} d_{X_i}(y_i, z_i).$$

The **metric space ultraproduct** along $\mathcal{U}$ of the family $(X_i)_{i\in I}$ centered at $(x_i)_{i\in I}$ is the metric quotient of the pseudometric space $(\ell^\infty(X_i, x_i, I), d)$. We denote it $\left(\prod_{i\in I}(X_i, x_i)\right)_{\mathcal{U}}$.

Remark 4.3. Any ultraproduct of complete metric spaces is easily seen to be complete.

In a normed space, the origin is a canonical choice for a distinguished point. The ultraproduct of a family of normed spaces, centered at the family of origins, comes with a natural structure of normed space. If all the normed spaces are Banach spaces, then by the above remark, their ultraproduct also is a Banach space. This normed space then induces a structure of affine normed space on the ultraproduct of normed spaces centered in an arbitrary family of points. Moreover, the choice of distinguished points does not matter too much.

Proposition 4.4. Let $(E_i)_{i\in I}$ be a family of normed spaces. Let $(x_i)_{i\in I}$ and $(x'_i)_{i\in I}$ be two families of distinguished points. Let $\mathcal{U}$ be an ultrafilter on I. Then the ultraproducts of $\left(\prod_{i\in I}(E_i, x_i)_{i\in I}\right)_{\mathcal{U}}$ and $\left(\prod_{i\in I}(E_i, x'_i)_{i\in I}\right)_{\mathcal{U}}$ are affinely isomorphic and isometric.

Proof. Consider the linear translation $(y_i)_{i\in I} \mapsto (y_i - x_i + x'_i)_{i\in I}$ in the product $\prod_{i\in I} E_i$. It sends $\ell^\infty(X_i, x_i, I)$ to $\ell^\infty(X_i, x'_i, I)$ and preserves the pseudometric. Hence, it defines an isometry between the two ultraproducts.

Moreover, since the isometry comes from a translation, the two ultraproducts are affinely isomorphic. □

When all the normed spaces E_i's are equal, say to a Banach space E, an ultraproduct of the family $(E_i)_{i\in I}$ centered at the family of origins is a Banach space, called a **Banach space ultrapower** of E.

5. Superreflexive Banach spaces

A Banach space E is said to be **superreflexive** if every Banach space ultrapower of E is reflexive. Enflo exhibited a characterization of superreflexivity in terms of convexity properties ([E1, corollary 3]): a Banach space is superreflexive if and only if it admits an equivalent norm that is uniformly convex.

Remark 5.1. In Enflo's theorem, superreflexivity is defined a bit differently; see [HM, theorem 2.3] and [S4, proposition 1.1] for the equivalence of the two definitions.

Definition 5.2. A Banach space $(E, \|\cdot\|)$ is **uniformly convex** if for every $\epsilon > 0$, there exists $\delta > 0$ such that for every x, y in E with $\|x\| = 1$, $\|y\| = 1$, one has

$$\|x - y\| \geqslant \epsilon \Rightarrow \left\|\frac{x+y}{2}\right\| \leqslant 1 - \delta.$$

In other words, a Banach space is uniformly convex if and only if its unit ball is strictly convex, this in a uniform way.

Examples 5.3. The following Banach spaces are superreflexive.

- Hilbert spaces.
- L^p spaces, for $1 < p < \infty$. This is a consequence of the Clarkson inequalities ([C, theorem 2]).

Superreflexivity is preserved under taking ℓ^2-type sums (the key argument is the Minkowski inequality).

Proposition 5.4. (Day, [D, theorem 2]) Let E be a superreflexive Banach space and X an arbitrary set. Then the Banach space $\ell^2(X, E)$ is superreflexive too.

Though uniform convexity is more workable a notion, it is intrinsically metric and it is not stable under Banach space isomorphisms, whereas superreflexivity is. Hence, since both uniform and coarse structures are invariant under isomorphisms, we state the embeddings results with superreflexivity rather than with uniform convexity.

The result we will present the proof of in the next two sections is the following.

Theorem 5.5. (Pestov) The Urysohn space does not admit any uniform embedding into a superreflexive Banach space.

Remark 5.6. Just around the same time Pestov's paper was published, a stronger result was proven by Kalton in [K1]: that the space c_0 does not admit any uniform embedding into a *reflexive* Banach space. Since c_0 is a Polish space, it embeds isometrically into the Urysohn space, so it follows that $\mathbb{U}$ does not admit any uniform embedding into a reflexive Banach space either. Still, Pestov's proof is based on very different techniques and is worth presenting.

Superreflexivity is a strengthening of reflexivity that invites ultraproducts constructions. The next section contains the main argument of Pestov's proof, an ultraproduct construction designed to *smoothen* actions on Banach spaces.

6. Averaging distances

Theorem 6.1. Let G be a locally finite group acting by isometries on a metric space X. Suppose that X admits a mapping φ into a normed space E such that for some functions ρ_1, $\rho_2 : \mathbb{R}_+ \to \mathbb{R}_+$:

$$\rho_1(d_X(x,x')) \leqslant \|\varphi(x) - \varphi(x')\| \leqslant \rho_2(d_X(x,x')).$$

Then there is a map ψ of X into a Banach space ultrapower of some $\ell^2(\mathcal{U}, E)$, satisfying the same inequalities

$$\rho_1(d_X(x,x')) \leqslant \|\psi(x) - \psi(x')\| \leqslant \rho_2(d_X(x,x')), \tag{1}$$

and such that the action of G on $\psi(X)$ extends to an action of G by affine isometries on the affine span of $\psi(X)$.

Proof. Let Ξ be the set of all finite subgroups of G. For every finite subgroup F in Ξ, we define a map $\psi_F : X \to \ell^2(F, E)$ by

$$\psi_F(x)(f) = \frac{1}{\sqrt{\operatorname{Card} F}} \varphi(f^{-1} \cdot x),$$

for every x in X and f in F.

Since G acts on X by isometries, the maps ψ_F satisfy the inequalities (1):

$$\rho_1(d_X(x,x')) \leqslant \|\psi_F(x) - \psi_F(x')\| \leqslant \rho_2(d_X(x,x')).$$

Indeed, let x and x' be two elements of X. Then we have:

$$\begin{aligned}
\|\psi_F(x) - \psi_F(x')\|_2 &= \left(\sum_{f \in F} \|\psi_F(x)(f) - \psi_F(x')(f)\|_E^2 \right)^{1/2} \\
&= \left(\frac{1}{\operatorname{Card} F} \sum_{f \in F} \|\varphi(f^{-1} \cdot x) - \varphi(f^{-1} \cdot x')\|_E^2 \right)^{1/2} \\
&\leqslant \left(\frac{1}{\operatorname{Card} F} \sum_{f \in F} \rho_2^2(d_X(f^{-1} \cdot x, f^{-1} \cdot x')) \right)^{1/2} \\
&= \left(\frac{1}{\operatorname{Card} F} \sum_{f \in F} \rho_2^2(d_X(x,x')) \right)^{1/2} \\
&= \rho_2(d_X(x,x'))
\end{aligned}$$

and similarly

$$\begin{aligned}\|\psi_F(x)-\psi_F(x')\|_2 &= \left(\frac{1}{\operatorname{Card} F}\sum_{f\in F}\|\varphi(f^{-1}\cdot x)-\varphi(f^{-1}\cdot x')\|_E^2\right)^{1/2}\\ &\geqslant \left(\frac{1}{\operatorname{Card} F}\sum_{f\in F}\rho_1^2(d_X(f^{-1}\cdot x, f^{-1}\cdot x'))\right)^{1/2}\\ &= \left(\frac{1}{\operatorname{Card} F}\sum_{f\in F}\rho_1^2(d_X(x,x'))\right)^{1/2}\\ &= \rho_1(d_X(x,x')).\end{aligned}$$

We would like to find a map that is compatible with the action of G. The group F acts on $\ell^2(F,E)$ by isometries, via the left regular representation: for $r\in\ell^2(F,E)$ and f, g in F, we define

$$^g r(f) = r(g^{-1}f).$$

Then the map ψ_F becomes F-equivariant:

$$\begin{aligned}{}^g(\psi_F(x))(f) &= \psi_F(x)(g^{-1}f)\\ &= \frac{1}{\sqrt{\operatorname{Card} F}}\varphi(f^{-1}g\cdot x)\\ &= \psi_F(g\cdot x)(f).\end{aligned}$$

Now we are average out all the maps ψ_F's. Choose an ultrafilter $\mathcal{U}$ on Ξ with the property that for each F in Ξ, the set $\{H\in\Xi : F\subseteq H\}$ is in $\mathcal{U}$. The local finiteness of the group G guarantees that such an ultrafilter exists.

Choose a point x^* in X. This yields distinguished points $\psi_F(x^*)$ in the $\ell^2(F,E)$'s. More precisely, let

$$V=\left(\prod_{F\in\Xi}(\ell^2(F,E),\psi_F(x^*))\right)_{\mathcal{U}}$$

be the ultraproduct of the spaces $\ell^2(F,E)$ along $\mathcal{U}$ centered at the family $(\psi_F(x^*))_{F\in\Xi}$.

We now prove that for every x in X, the family $(\psi_F(x))_{F\in\Xi}$ is at finite distance from the distinguished family $(\psi_F(x^*))_{F\in\Xi}$, hence its class defines an element of V. Let x be an element of X.

$$\begin{aligned}\sup_{F\in\Xi}\|\psi_F(x)-\psi_F(x^*)\| &\leqslant \sup_{F\in\Xi}\rho_2(d_X(x,x^*))\\ &= \rho_2(d_X(x,x^*)).\end{aligned}$$

This implies we can define a map $\psi : X\to V$ by

$$\psi(x) = [(\psi_F)_{F\in\Xi}]_{\mathcal{U}}\,.$$

Moreover, the action of G on the space V is well-defined: let g be an element of G. Since G is locally finite, the subgroup of G generated by g is finite, hence in Ξ. We chose the ultrafilter $\mathcal{U}$ in such a way that the set of all F in Ξ that contain $\langle g\rangle$ is in $\mathcal{U}$. From this, it follows that g acts on $\ell^2(F,E)$ for $\mathcal{U}$-every F in Ξ.

Since the action of F on each $\ell^2(F,E)$ is an action by isometries, so is the action of G on V. For this action, the map ψ is G-equivariant as desired.

It remains to identify the ultraproduct V with a Banach space ultrapower of $\ell^2(\mathcal{U},E)$. First, note that $\ell^2(\mathcal{U},E)$ contains every $\ell^2(F,E)$ as a normed space (this embedding is not canonical;

this is just because F is finite and $\mathcal{U}$ is bigger). Thus, V is contained in a suitable ultraproduct of $\ell^2(\mathcal{U}, E)$, which is isometrically and affinely isomorphic to the corresponding Banach space ultrapower of $\ell^2(\mathcal{U}, E)$ by proposition 4.4. □

7. Obstruction to a uniform embedding

Theorem 7.1. The Urysohn space $\mathbb{U}$ cannot be uniformly embedded into a superreflexive Banach space.

Proof. Suppose it can and let $\varphi : \mathbb{U} \to E$ be a uniform embedding of $\mathbb{U}$ into a superreflexive Banach space E. Let also ρ_1 and ρ_2 be two decreasing functions from $\mathbb{R}_+$ to $\mathbb{R}_+$, with $0 < \rho_1 \leqslant \rho_2$ and $\lim_{r \to 0} \rho_2(r) = 0$, witnessing that φ is a uniform embedding: such that for all x, x' in $\mathbb{U}$, one has

$$\rho_1(d_{\mathbb{U}}(x, x')) \leqslant \|\varphi(x), \varphi(x')\|_E \leqslant \rho_2(d_{\mathbb{U}}(x, x')).$$

Let G be a dense locally finite subgroup of $\mathrm{Iso}(\mathbb{U})$ (such a subgroup exists by proposition 3.3). By proposition 6.1, there exists a mapping ψ of $\mathbb{U}$ into a Banach space ultrapower V of $\ell^2(\mathcal{U}, E)$ such that for all x, x' in $\mathbb{U}$, one has

$$\rho_1(d_{\mathbb{U}}(x, x')) \leqslant \|\psi(x), \psi(x')\|_V \leqslant \rho_2(d_{\mathbb{U}}(x, x')), \tag{2}$$

and such that the action of G extends to an action by affine isometries on the affine span S of $\psi(\mathbb{U})$ in V, making ψ G-equivariant.

Note that V is reflexive as an ultrapower of the superreflexive space $\ell^2(\mathcal{U}, E)$, as to proposition 5.4.

The inequalities (2) guarantee that ψ is a uniform isomorphism on its image. In particular, ψ is a homeomorphism. So the topology on G of pointwise convergence on $\mathbb{U}$ coincides with the topology of pointwise convergence on $\psi(\mathbb{U})$, and consequently, on S as G acts by affine isometries.

Moreover, since ψ is a uniformly continuous, so is the representation of G on S. Thus, by density of G in $\mathrm{Iso}(\mathbb{U})$, the action of G extends to a uniformly continuous action of $\mathrm{Iso}(\mathbb{U})$ on S for which the map ψ remains equivariant. It follows that the representation of $\mathrm{Iso}(\mathbb{U})$ on S is faithful: if g and h are isometries such that for all x in $\mathbb{U}$, one has $g \cdot \psi(x) = h \cdot \psi(x)$, then by equivariance, one has $\psi(g \cdot x) = \psi(h \cdot x)$ for all x in $\mathbb{U}$. But since ψ is an isomorphism, this implies that for all x in $\mathbb{U}$, one has $g \cdot x = h \cdot x$, hence $g = h$.

Write this affine representation of $\mathrm{Iso}(\mathbb{U})$ on S is a continuous homomorphism from $\mathrm{Iso}(\mathbb{U})$ to the group $\mathrm{Iso}(S) = \mathrm{LIso}(S) \ltimes S_+$, where S_+ is the additive group of S (group of translations) and $\mathrm{LIso}(S)$ the group of linear isometries of S. Let also π denote the standard (continuous) projection from $\mathrm{LIso}(S) \ltimes S_+$ onto $\mathrm{LIso}(S)$.

Now recall that the group $\mathrm{Iso}(\mathbb{U})$ is a universal Polish group (Uspenskij [U2], see section 2). In particular, it contains $\mathrm{Homeo}_+[0, 1]$ as a topological subgroup. Therefore, we have a faithful continuous affine representation of the group $\mathrm{Homeo}_+[0, 1]$ in the reflexive Banach space V.

But Megrelishvili proved in [M1] that the only continuous representation of $\mathrm{Homeo}_+[0, 1]$ by linear isometries on a reflexive Banach space is the trivial representation (see theorem 2.8). Therefore, the linear part of the restriction of π to $\mathrm{Homeo}_+[0, 1]$ is trivial. $\mathrm{Homeo}_+[0, 1]$ then has to act by translations, but by faithfulness of the representation, this implies that $\mathrm{Homeo}_+[0, 1]$ is abelian, a contradiction. □

8. Concluding remarks

Let us mention which (non-)embeddability properties of the Urysohn space remain when we relax or sharpen our notion of embedding.

8.1. **Coarse embeddability.** Whereas the uniform structure gives the local behavior of metric spaces, the coarse structure, or *large-scale structure*, of a metric space describes its geometry *at infinity*.

A map $f : X \to Y$ is a **coarse embedding** of X into Y if there exist two non-decreasing unbounded functions ρ_1 and ρ_2 from $\mathbb{R}_+$ to $\mathbb{R}_+$, with $0 < \rho_1 \leqslant \rho_2$, such that for all x, x' in X, one has

$$\rho_1(d_X(x,x')) \leqslant d_Y(f(x), f(x')) \leqslant \rho_2(d_X(x,x')).$$

In particular, for a fixed x' in X, the distance $d_X(x,x')$ tends to infinity if and only if $d_X(f(x), f(x'))$ does. Note that a coarse embedding is not necessarily continuous.

Pestov also applies the techniques of theorem 6.1 to coarse embeddings to prove that the Urysohn space does not admit any coarse embedding into a superreflexive Banach space either. The proof is way more technical though[3]. Moreover, it is based on a strengthening of theorem 6.1 ([P, corollary 4.4]), the proof of which I did not understand. It states that if the locally finite group G acts *almost* transitively on the space X, then the image $\psi(X)$ we build is a *metric transform* of X, meaning that the distance $\|\psi(x) - \psi(x')\|$ depends only on $d(x,x')$.

In [K1], Kalton proved a stronger result: the Urysohn space does not even admit any coarse embedding into a reflexive Banach space. It follows from the same result for the space c_0 (see also remark 5.6).

8.2. **Isometric embeddability.** We could also simply consider isometric embeddings of the Urysohn space, which are a very special case of uniform embeddings. However, this proves to be too restrictive: there is only one way to embed the Urysohn space isometrically into a Banach space. Whenever $\mathbb{U}$ embeds isometrically into a Banach space, then the span of its image is the *Holmes space* ([H1, theorem 6]).

In conclusion, it is quite hard to embed $\mathbb{U}$ nicely into Banach spaces!

References

[B] S. Banach, *Théorie des opérations linéaires*, Chelsea Publishing Co., New York, 1955.

[C] J. A. Clarkson, *Uniformly convex spaces*, Trans. Amer. Math. Soc. **40** (1936), no. 3, 396–414.

[D] M. M. Day, *Some more uniformly convex spaces*, Bull. Amer. Math. Soc. **47** (1941), 504–507.

[E1] P. Enflo, *Banach spaces which can be given an equivalent uniformly convex norm*, Proceedings of the International Symposium on Partial Differential Equations and the Geometry of Normed Linear Spaces (Jerusalem, 1972), 1972, pp. 281–288 (1973).

[E2] R. Engelking, *General topology*, Second, Sigma Series in Pure Mathematics, vol. 6, Heldermann Verlag, Berlin, 1989. Translated from the Polish by the author.

[G] S. Gao, *Invariant descriptive set theory*, Pure and Applied Mathematics (Boca Raton), vol. 293, CRC Press, Boca Raton, FL, 2009.

[GK] S. Gao and A. S. Kechris, *On the classification of Polish metric spaces up to isometry*, Memoirs of Amer. Math. Soc. **766** (2003).

[H1] M. R. Holmes, *The Urysohn space embeds in Banach spaces in just one way*, Topology Appl. **155** (2008), no. 14, 1479–1482.

[H2] E. Hrushovski, *Extending partial isomorphisms of graphs*, Combinatorica **12** (1992), no. 4, 411–416.

[HM] C. W. Henson and L. C. Moore Jr., *Subspaces of the nonstandard hull of a normed space*, Trans. Amer. Math. Soc. **197** (1974), 131–143.

[K1] N. J. Kalton, *Coarse and uniform embeddings into reflexive spaces*, Q. J. Math. **58** (2007), no. 3, 393–414.

[K2] M. Katětov, *On universal metric spaces*, General topology and its relations to modern analysis and algebra, VI (Prague, 1986), 1988, pp. 323–330.

[M1] M. G. Megrelishvili, *Every semitopological semigroup compactification of the group $H_+[0,1]$ is trivial*, Semigroup Forum **63** (2001), no. 3, 357–370.

[3] "Now, one can verify, by considering 17 separate cases, that..."!

[M2] J. Melleray, *Some geometric and dynamical properties of the Urysohn space*, Topology Appl. **155** (2008), no. 14, 1531–1560.

[P] V. G. Pestov, *A theorem of Hrushovski-Solecki-Vershik applied to uniform and coarse embeddings of the Urysohn metric space*, Topology Appl. **155** (2008), no. 14, 1561–1575.

[S1] W. Sierpiński, *Sur un espace métrique séparable universel*, Fund. Math. **33** (1945), 115–122.

[S2] S. Solecki, *Extending partial isometries*, Israel J. Math. **150** (2005), 315–331.

[S3] ______, *Notes on a strengthening of the herwig-lascar extension theorem*, 2009.

[S4] J. Stern, *Ultrapowers and local properties of Banach spaces*, Trans. Amer. Math. Soc. **240** (1978), 231–252.

[U1] P. S. Urysohn, *Sur un espace métrique universel*, Bull. Sci. Math. **51** (1927), 43–64 and 74–96.

[U2] V. V. Uspenskij, *On the group of isometries of the Urysohn universal metric space*, Comment. Math. Univ. Carolin. **31** (1990), no. 1, 181–182.

[V] A. M. Vershik, *Globalization of the partial isometries of metric spaces and local approximation of the group of isometries of Urysohn space*, Topology Appl. **155** (2008), no. 14, 1618–1626.