PROJECT ON A SOFT INTRODUCTION TO CARDINALITY THEORY

In this project we have a brief look at the problem of comparing the size of sets. Size comparison will be done via the concept of bijective maps.

1. Comparing the size of sets

Definition 1 (Equinumerability)**.** *Let X and Y be sets. We say that X and Y are* equinumerous *(or equipotent) if there exists a bijection from X onto Y. If X is equinumerous to Y we write $X \approx Y$.*

Exercise 1. *(3 points) Let X, Y, Z be sets. Show that*

(1) *(1 point) $X \approx X$.*
(2) *(1 point) If $X \approx Y$ then $Y \approx X$.*
(3) *(1 point) If $X \approx Y$ and $Y \approx Z$ then $X \approx Z$.*

Exercise 1 suggests that $\approx$ seems to have all the attributes of an equivalence relation that would be defined on the collection of all sets ... but there is an issue with this last statement since the collection of all sets CANNOT be a set (this is Russell's Paradox), and we only introduced relations on sets! This is a delicate point that can only be overcome using a rigorous axiomatic approach for set theory instead of the elementary naive approach that we have undertaken.

In the next exercise we compare the size of a set to the size of its power set.

Exercise 2. *(7 points) Let X be a set.*

(1) *(3 point) Show that there exists an injection from X into $\mathscr{P}(X)$.*
(2) *(4 point) Show that there does not exist a surjection from X onto $\mathscr{P}(X)$.*

We write $X \preceq Y$ if there is an injection from X into Y, and $X \prec Y$ if $X \preceq Y$ and $X \not\approx Y$. Considering $X = \mathbb{N}$, Exercise 2 tells you that $\mathbb{N} \prec \mathscr{P}(\mathbb{N})$. It can be shown that $\mathscr{P}(\mathbb{N}) \approx \mathbb{R}$ and hence $\mathbb{N} \prec \mathbb{R}$. The Continuum Hypothesis states that there does not exist a set X such that $\mathbb{Q} \prec X \prec \mathbb{R}$. More informally, there is no set whose cardinality lies strictly between the cardinality of the "discrete" set $\mathbb{N}$ and the "continuous" set $\mathbb{R}$.