PROJECT ON A SOFT INTRODUCTION TO MEASURE THEORY

In this project you will study collections of subsets (of an ambient set) that satisfy certain stability properties with respect to some elementary set-theoretic operations that you have learned and studied in class. Such a collection is called a *σ-algebra* and is the fundamental structure of an abstract mathematical theory called measure theory. Measure theory has found many applications. In particular, it provides the foundational mathematical framework for modern probability theory. Your knowledge of set theory learned in MATH 300 is sufficient to be able to study the basic properties of these σ-algebras and functions between them.

1. Measure spaces

Definition 1 (σ-algebra)**.** *Let X be a set. A collection $\mathscr{M}$ of subsets of X is called a* σ-algebra *if the following properties are satisfied:*

(Σ_1) $X \in \mathscr{M}$.
(Σ_2) *For all $A \in \mathscr{M}$ we have $X \setminus A \in \mathscr{M}$ (stability under complementation).*
(Σ_3) *For all countable collection $\{A_n\}_{n\in\mathbb{N}}$ of elements in $\mathscr{M}$ (i.e., $A_n \in \mathscr{M}$ for all $n \in \mathbb{N}$) we have $\bigcup_{n=1}^{\infty} A_n \in \mathscr{M}$ (stability under countable unions).*

A set X equipped with a σ-algebra $\mathscr{M}$ is called a *measure space* and the sets in $\mathscr{M}$ are called *measurable sets*. In Exercise 1 and 2 we describe some simple examples of σ-algebras.

Exercise 1. *(6 points) Let X be a set.*

(1) *(3 points) Consider $\mathscr{M}_{trivial} = \{\emptyset, X\}$. Prove that $\mathscr{M}_{trivial}$ is a σ-algebra on X.*
(2) *(3 points) Consider $\mathscr{M}_{discrete} = \mathscr{P}(X)$. Is $\mathscr{M}_{discrete}$ a σ-algebra on X? Justify briefly your answer.*

Exercise 2. *(6 points) Let X be a set and A be a non-empty proper subset of X (i.e., $\emptyset \subset A \subset X$). Show that $\mathscr{M} = \{\emptyset, A, X \setminus A, X\}$ is a σ-algebra on X.*

In the next exercise you are asked to prove that a σ-algebra is also stable under finite unions. This result follows by combining in a clever way property (Σ_1) with (Σ_3).

Exercise 3. *(8 points) Let X be a set and $\mathscr{M}$ be a σ-algebra on X. Show that $\mathscr{M}$ is stable under finite unions.*

Hint: Formally, you must show that for all $k \geqslant 1$, and for all finite collection $\{A_n\}_{n=1}^{k}$ of elements in $\mathscr{M}$ (*i.e.*, $A_n \in \mathscr{M}$ for all $1 \leqslant n \leqslant k$) we have $\bigcup_{n=1}^{k} A_n \in \mathscr{M}$.

Using de Morgan's laws we can prove that a σ-algebra is also stable under countable intersections.

Exercise 4. *(6 points) Let X be a set and $\mathscr{M}$ be a σ-algebra on X. Show that $\mathscr{M}$ is stable under countable intersections.*

As in the case of unions, stability by countable intersections implies stability by finite intersections.

Exercise 5. *(6 points) Let X be a set and $\mathscr{M}$ be a σ-algebra on X. Show that $\mathscr{M}$ is stable under finite intersections.*

Exercise 6. *(4 points) Let X be a set and $\mathscr{M}$ be a σ-algebra on X. Show that $\mathscr{M}$ is stable under set differences.*

Exercise 7. *(4 points) Let X be a set. The symmetric difference between two subsets A and B of X, is defined as the set*

$$A \triangle B = \{x \in X \colon [(x \in A) \wedge (x \notin B)] \vee [(x \in B) \wedge (x \notin A)]\}.$$

If $\mathscr{M}$ is a σ-algebra on X, show that $\mathscr{M}$ is stable under symmetric differences.

Hint: Formally, you must show that if $A, B \in \mathscr{M}$ then $A \triangle B \in \mathscr{M}$. You could rewrite $A \triangle B$ in terms of intersection, union, and complementation.

2. Creating new σ-algebras out of old ones: the pull-back procedure

Let $f\colon X \to Y$ be a function and let $\mathscr{C} \subseteq \mathscr{P}(Y)$. The collection of sets $\{f^{-1}(A)\colon A \in \mathscr{C}\}$ is a collection of subsets of $\mathscr{P}(X)$ which we will simply denote by $f^{-1}(\mathscr{C})$. The next exercise shows that if Y is equipped with a σ-algebra then there is a natural way to create a σ-algebra on X using the function f via the inverse image (this σ-algebra is called the pull-back algebra). This pull-back procedure crucially uses the Hausdorff formulas for inverse images.

Exercise 8. *(5 points) Let $f\colon X \to Y$ be a function and let $\mathscr{M} \subseteq \mathscr{P}(Y)$ be a σ-algebra on Y. Show that $f^{-1}(\mathscr{M}) = \{f^{-1}(A)\colon A \in \mathscr{M}\}$ is a σ-algebra on X.*