REAL ANALYSIS MATH 608
HOMEWORK #10

Problem 1. *Let $T \in L(H)$ where H is a Hilbert space.*

(1) *Show that there is a unique operator $T^* \in L(H)$ such that $< Tx, y >=< x, T^*y >$ for all $x, y \in H$.*
(2) *Show that $Ker(T) = T^*(H)^{\perp}$.*
(3) *Show that T is unitary if and only if $T^{-1} = T^*$.*
(4) *Show that a bounded linear map P is an orthogonal projection if and only if $P^2 = P = P^*$.*

Hint: For (1) use the representation theorem for Hilbert spaces and the adjoint of an arbitrary operator between Banach spaces.

Problem 2 (Reflexivity of L_p-spaces)**.**

(1) *Let X, Y be Banach spaces. Show that if $T : X \to Y$ is a surjective isometry then the dual operator $T^* : Y^* \to X^*$ is a surjective isometry.*
(2) *Show that, for every $p \in (1, \infty)$ and measure μ, $L_p(\mu)$ is reflexive.*

Hint: For (2) use (1) and the representations theorems for $L_p(\mu)$ spaces to define a surjective isometry between $L_p(\mu)$ and $L_p(\mu)^{**}$, and verufy that this map coincides with the canonical isometric embedding of $L_p(\mu)$ into $L_p(\mu)^{**}$.

Problem 3.

(1) *Let $1 < p < q < \infty$ and $f \in L_{p,\infty}(\mu) \cap L_{q,\infty}(\mu)$. Show that $f \in L_r(\mu)$ for all $r \in (p, q)$ and if $\frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{q}$*

$$\|f\|_r \leqslant \left(\frac{r}{r-p} + \frac{r}{q-r} \right)^{\frac{1}{r}} \|f\|_{p,\infty}^{1-\theta} \cdot \|f\|_{q,\infty}^{\theta}.$$

(2) *Deduce from assertion (1) the case $p_0 = p_1 = p \in (1, \infty)$ and $p \leqslant q_0 < q_1 < \infty$ of the Marcinkiewicz interpolation theorem.*
(3) *(Bonus question) Did you have to use the sublinearity of the map T and the condition $p \leqslant q_0 < q_1 < \infty$ in (2)?*

Hint: For (1) use that $\|f\|_r^r = r \int_0^\infty t^{r-1} \mu\{|f| > t\} dt$ and split the integral at $\beta = \|f\|_{q,\infty}^{q/(q-p)} \|f\|_{p,\infty}^{-p/(q-p)}$.