REAL ANALYSIS MATH 608
HOMEWORK #10

Problem 1. *Let $T \in L(H)$ where H is a Hilbert space.*

1. *Show that there is a unique operator $T^* \in L(H)$ such that $< Tx, y >=< x, T^*y >$ for all $x, y \in H$.*
2. *Show that $Ker(T) = T^*(H)^\perp$.*
3. *Recall that T is unitary iff T is a surjective linear map that preserves the scalar product iff T is an onto isometry.*
   Show that T is unitary if and only if T is invertible and $T^{-1} = T^*$, i.e. $TT^* = T^*T = I$.
4. *Show that T is unitary if and only if T^* is unitary.*
5. *Show that a bounded linear map P is an orthogonal projection if and only if $P^2 = P = P^*$.*

Hint: For (1) use the representation theorem for Hilbert spaces and the adjoint of an arbitrary operator between Banach spaces.

Solution.

1. Uniqueness is straightforward.
   Given $y \in H$, consider the map $\phi_y\colon\ x \in H \to< Px, y >\in \mathbb{F}$. One can easily verify that ϕ_y is a bounded linear form and by Riesz representation theorem there is a unique $z_y \in H$ such that $\phi_y(x) =< x, z_y >$, for all $x \in H$. If we let $P^*y = z_y$ (which is well-defined by uniqueness), then for all $x, y \in H$ one has $< Px, y >=< x, P^*y >$.
   Another way to prove existence is as follows. Let $T^* \stackrel{\text{def}}{=} \Phi^{-1} \circ T^d \circ \Phi$ where T^d is the adjoint operator (in the Banach space sense) and $\Phi\colon\ H \to H^*$ is the anti-linear surjective isometry from Riesz representation theorem, i.e. for all $y^* \in H^*$ and $h \in H$, $y^*(h) =< h, \Phi^{-1}(y^*) >$. Then, $T^*\colon\ H \to H$ is clearly linear and bounded, and it remains to observe that $< x, T^*y >=< x, \Phi^{-1}T^d\Phi y >= (T^d\Phi)(y)(x) = T^d(\Phi(y)(x) = \Phi(y)(Tx) =< Tx, y >$.
2. Since for all $x, h \in H$, $< x, T^*(h) >=< Tx, h >$ the conclusion immediately follows. Indeed, if $x \in \ker(T)$ then $< x, T^*(h) >=< 0, h >= 0$ and hence $\ker(T) \subset T^*(H)^\perp$, and if $x \in T^*(H)^\perp$ then $0 =< Tx, h >$ and $Tx \in H^\perp = \{0\}$.
3. If T is invertible and $T^{-1} = T^*$, then T is surjective by assumption and for all $x, y \in H$, $< Tx, Ty >=< x, T^*Ty >=< x, y >$, and thus T is unitary.
   Assume now that T is unitary. Then $< x, y >=< Tx, Ty >=< x, T^*Ty >$ for all $x, y \in H$ and thus $T^*T = I$, i.e. T is left invertible. But we know that T is invertible and hence $T^{-1} = T^*$.
4. Assume that T is unitary. It is easy to see by uniqueness of the adjoint that $I^* = I$ ($< Ix, y >=< x, Iy >$) and $(AB)^* = B^*A^*$ ($< ABx, y >=< Bx, A^*y >=< x, B^*A^*y >$). Since by assumption T^* is invertible and $TT^* = I$, after taking adjoints we have that $(TT^*)^* = T^{**}T^* = I^* = I$, and hence $(T^*)^{-1} = (T^*)^*$, i.e. T^* is unitary.
   Another proof is as follows. Since T is unitary it preserves the scalar product, and one has $< T^*x, T^*y >=< TT^*, TT^*y >=< x, y >$, and T preserves the scalar product. Moreover, since T^* is surjective (because invertible when T is unitary) it follows that T^* is unitary.
   Yet another proof uses the definition of T^* in terms of the Banach space adjoint and goes as follows: If T is unitary, we know that T^* is invertible and thus surjective. It remains to show that T^*

preserve the scalar product. But

$$\begin{aligned}< T^*x, T^*y >_H &=< \Phi^{-1}T^d\Phi x, \Phi^{-1}T^d\Phi y >_H \text{ (definition of the adjoint)}\\ &=< T^d\Phi x, T^d\Phi y >_{H^*} \text{ (definition of } < \cdot,\cdot >_{H^*})\\ &=< \Phi x, \Phi y >_{H^*} \ (T^d\colon H^* \to H^* \text{ is an isometry since } T\colon H \to H \text{ is})\\ &=< x, y >_H \text{ (definition of } < \cdot,\cdot >_{H^*}).\end{aligned}$$

For the converse, observe that

$$< T^*x, y >= \overline{< y, T^*x >} = \overline{< Ty, x >} =< x, Ty >,$$

and it follows from uniqueness of the adjoint that $(T^*)^* = T$. Therefore, if T^* is unitary, then by the previous implication $(T^*)^* = T$ is unitary.

We could also say that since T^* is unitary it preserves the scalar product and $T^*T^{**} = T^{**}T^* = I$, and one has $< Tx, Ty >=< T^*Tx, T^*Ty >=< T^*T^{**}x, T^*T^{**}y >=< x, y >$, and T preserves the scalar product. Moreover, since T^* is surjective (because invertible when T is unitary) it follows that T^* is unitary.

Feel free to find other proofs.

(5) If $P^2 = P = P^*$ then by (2) $\ker(P) = P(H)^\perp$, i.e. $\ker(P) \perp P(H)$ and P is an orthogonal projection. Assume now that P is an orthogonal projection, then $P(H) \perp \ker(P)$, and

$$< Px, y >=< Px, y - Py > + < Px, Py >=< Px, Py >=< Px - x, Py > + < x, Py >=< x, Py >,$$

and by uniqueness of the adjoint we conclude that $P^* = P$.

□

Problem 2 (Reflexivity of L_p-spaces)**.**

(1) Let X, Y be Banach spaces. Show that if $T\colon X \to Y$ is a surjective isometry then the dual operator $T^*\colon Y^* \to X^*$ is a surjective isometry.

(2) Show that, for every $p \in (1,\infty)$ and measure μ, $L_p(\mu)$ is reflexive.

Hint: For (2) use (1) and the representations theorems for $L_p(\mu)$ spaces to define a surjective isometry between $L_p(\mu)$ and $L_p(\mu)^{**}$, and verify that this map coincides with the canonical isometric embedding of $L_p(\mu)$ into $L_p(\mu)^{**}$.

Solution.

(1) T^* is clearly linear and $|T^*(y^*)(x)| = |y^*(Tx)| \leqslant \|y^*\| \cdot \|T\| \cdot \|x\|$ and hence $\|T^*\| \leqslant \|T\|$. Given $\varepsilon > 0$ and $x \in X$ with $\|x\| = 1$ such that $\|Tx\| \geqslant (1-\varepsilon)\|T\|$. By Hahn-Banach theorem we can pick $y^* \in Y^*$ such that $\|y^*\| = 1$ and $y^*(Tx) = \|Tx\|$, then $|T^*(y^*)(x)| = |y^*(Tx)| = \|Tx\| \geqslant (1-\varepsilon)\|T\|$, and hence $\|T^*\| \geqslant (1-\varepsilon)\|T\|$. Letting $\varepsilon \to 0$ we conclude that $\|T\| = \|T^*\|$. Now, since $T(B_X) = B_Y$ as T is an onto isometry we have

$$\|T^*(y^*)\| = \sup_{x\in B_X} \|T^*(y^*)(x)\| = \sup_{x\in B_X} |y^*(T(x))| = \sup_{y\in T(B_X)} \|y^*(y)\| = \sup_{y\in B_Y} \|y^*(y)\| = \|y^*\|,$$

and T^* is an isometry. To show surjectivity of T^*, let $x^* \in X^*$ and put $y^* = x^* \circ T^{-1}$. Then

$$T^*(y^*) = y^* \circ T = x^* \circ T^{-1} \circ T = x^*.$$

(2) Let $\frac{1}{p} + \frac{1}{q} = 1$ with $p, q \in (1,\infty)$, and $\Phi_p\colon L_q \to L_p^*$, $\Phi_q\colon L_p \to L_q^*$ be the surjective isometries given by Riesz representation theorem. We will show that $\delta = (\Phi_p^{-1})^* \circ \Phi_q$ where δ is the canonical isometric

embedding $\delta\colon L_p \to L_p^{**}$ defined by $\delta(f)(y^*) = y^*(f)$. Note that $(\Phi_p^{-1})^d \circ \Phi_q$ is clearly linear and surjective (by (1)). For any $f \in L_p$, and $y^* \in L_p^*$, it follows from the definition of the adjoint that

$$((\Phi_p^{-1})^* \circ \Phi_q)(f)(y^*) = (\Phi_p^{-1})^*(\Phi_q(f))(y^*) = \Phi_q(f)(\Phi_p^{-1}(y^*)),$$

and since by definition of Φ_q and Φ_p we have

$$\Phi_q(f)(\Phi_p^{-1}(y^*)) = \int f\Phi_p^{-1}(y^*)d\mu = y^*(f),$$

the conclusion follows.

□

Problem 3.

(1) Let $1 < p < q < \infty$ and $f \in L_{p,\infty}(\mu) \cap L_{q,\infty}(\mu)$. Show that $f \in L_r(\mu)$ for all $r \in (p,q)$ and if $\frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{q}$

$$\|f\|_r \leqslant \left(\frac{r}{r-p} + \frac{r}{q-r}\right)^{\frac{1}{r}} \|f\|_{p,\infty}^{1-\theta} \cdot \|f\|_{q,\infty}^{\theta}.$$

(2) Deduce from assertion (1) the case $p_0 = p_1 = p \in (1,\infty)$ and $p \leqslant q_0 < q_1 < \infty$ of the Marcinkiewicz interpolation theorem.

(3) (Bonus question) Did you have to use the sublinearity of the map T and the condition $p \leqslant q_0 < q_1 < \infty$ in (2)?

Hint: For (1) use that $\|f\|_r^r = r\int_0^\infty t^{r-1}\mu\{|f| > t\}dt$ and split the integral at $t_0 = \|f\|_{q,\infty}^{q/(q-p)}\|f\|_{p,\infty}^{-p/(q-p)}$.

Solution.

(1) Let f_* denote the distribution function of f, i.e. $f_*(t) \overset{\text{def}}{=} \mu(\{|f| > t\})$ and recall that by definition $t^s f_*(t) \leqslant \|f\|_{s,\infty}^s$ for all t, s. Then,

$$\begin{aligned}\|f\|_r^r &= r\int_0^\infty t^{r-1} f_*(t)dt \\ &\leqslant r\int_0^\infty t^{r-1}\min\left\{\frac{\|f\|_{p,\infty}^p}{t^p}, \frac{\|f\|_{q,\infty}^q}{t^q}\right\}dt.\end{aligned}$$

Observe now that $\frac{\|f\|_{p,\infty}^p}{t^p} \leqslant \frac{\|f\|_{q,\infty}^q}{t^q}$ if and only if $t \leqslant \left(\frac{\|f\|_{q,\infty}^q}{\|f\|_{p,\infty}^p}\right)^{\frac{1}{q-p}} := t_0$. Then,

$$\begin{aligned}\|f\|_r^r &\leqslant r\int_0^{t_0} t^{r-1}\min\left\{\frac{\|f\|_{p,\infty}^p}{t^p}, \frac{\|f\|_{q,\infty}^q}{t^q}\right\}dt + r\int_{t_0}^\infty t^{r-1}\min\left\{\frac{\|f\|_{p,\infty}^p}{t^p}, \frac{\|f\|_{q,\infty}^q}{t^q}\right\}dt \\ &\leqslant r\int_0^{t_0} t^{r-1-p}\|f\|_{p,\infty}^p + r\int_{t_0}^\infty t^{r-1-q}\|f\|_{q,\infty}^q \\ &= \frac{r}{r-p}\|f\|_{p,\infty}^p t_0^{r-p} + \frac{r}{q-r}\|f\|_{q,\infty}^q t_0^{r-q} (\text{ since } p < r < q) \\ &= \frac{r}{r-p}\|f\|_{p,\infty}^{p-p\frac{r-p}{q-p}} \cdot \|f\|_{q,\infty}^{q\frac{r-p}{q-p}} + \frac{r}{q-r}\|f\|_{q,\infty}^{q+q\frac{r-q}{q-p}} \cdot \|f\|_{q,\infty}^{-p\frac{r-q}{q-p}} (\text{ by definition of } t_0).\end{aligned}$$

An elementary computation shows that $r\theta = q\frac{p-r}{p-q}$ and $r(1-\theta) = p\frac{q-r}{q-p}$, but $p - p\frac{r-p}{q-p} = p(1-\frac{r-p}{q-p}) = p\frac{q-p-r+p}{q-p} = p\frac{q-r}{q-p}$, $q + q\frac{r-q}{q-p} = q\frac{q-p+r-q}{q-p} = q\frac{r-p}{q-p}$. Collecting terms and taking the r-th root gives the desired inequality.

(2) Assume that T maps L_p to $L_{q_0,\infty}$ and to $L_{q_1,\infty}$, then for all $f \in L_p$, $T(f) \in L_{q_0,\infty} \cap L_{q_1,\infty}$ and by (1) $T(f) \in L_{q_t}$ where $\frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}$ with

$$\|Tf\|_{q_t} \leqslant \left(\frac{q_t}{q_t - q_0} + \frac{q_t}{q_1 - q_t}\right)^{\frac{1}{q_t}} \|Tf\|_{q_0,\infty}^{1-t} \cdot \|Tf\|_{q_1,\infty}^{t} \leqslant A_t C_0^{1-t} \|f\|_p^{1-t} C_1^t \|f\|_p^t = A_t C_0^{1-t} C_1^t \|f\|_p,$$

where $A_t := \frac{q_t}{q_t-q_0} + \frac{q_t}{q_1-q_t}$, $C_0 := \|T : L_p \to L_{q_0,\infty}\|$ and $C_1 := \|T : L_p \to L_{q_1,\infty}\|$.

(3) No and no. These conditions are needed in the other cases.

□