REAL ANALYSIS MATH 608
HOMEWORK #1

Problem 1.

(1) Show that a topological space is T_1 if and only if every singleton is closed.
(2) Show that the cofinite topology on an infinite set X is T_1 but not T_2.

Solution. (1) If the space is T_1 then $\{x\} = \cap_{y\neq x}\{O_y\}^c$ where O_y is an open set that contains y but not x. For the converse, $\{x\}^c$ is an open set that contains any $y \neq x$ but not x.

(2) By definition every finite set is closed in the cofinite topology, so in particular singletons are closed. To show that it is not T_2, let $x \neq y \in X$ and consider an open set U containing x. Any set that is disjoint with U must then be included U^c which is finite. Thus any open set containing y and disjoint from U must be finite; this is not possible since X is infinite and any open set must be infinite.

□

Problem 2.

(1) Show that A is nowhere dense in $(X,\mathcal{T})$ if and only if for every nonempty open set U, $\overline{A}\cap U \neq U$.
(2) Interpret the statement in (1) in the relative topology.

Solution. (1) Let U be a nonempty open set. If A is nowhere dense then $(\overline{A}\cap U)^\circ = (\overline{A})^\circ \cap U^\circ = \emptyset \neq U$. For the converse, assume that A is not nowhere dense, then $U = (\overline{A})^\circ$ is open and nonempty, but $U \subset \overline{A}$ and thus $U \cap \overline{A} = U$.

(2) Since one can verify that $\overline{A}\cap U$ is equal to the closure of $A\cap U$ in the relative topology of U, item (1) says that for any non empty open set U, the subset $A \cap U$ of U is not dense in U for the relative topology on U.

□

Problem 3. *Show that a separable metric space is second countable.*

Solution. Let $\{x_n\}_{n=1}^\infty$ be a dense sequence in (X,d). Then $\{B(x_n,q)\}_{n\geqslant 1, q\in\mathbb{Q}}$ is a countable base for the metric topology. Indeed, if U is an open set that contains x then there is $r > 0$ such that $B(x,r) \subset U$. Since the open balls are nonempty open sets, by density of $\{x_n\}_{n=1}^\infty$ there must be an $x_{n_0} \in B(x, \frac{r}{4})$ and by density of the rational we can certainly find a positive rational $q \in (\frac{r}{4}, \frac{r}{2})$, and hence $x \in B(x_{n_0}, q) \subset B(x,r) \subset U$. □

Problem 4 (The Sorgenfrey line or upper limit topology)**.** *Let $\mathscr{C} \overset{\text{def}}{=} \{(a,b] : -\infty < a < b < \infty\}$.*

(1) Show that $\mathscr{C}$ is a base for a topology, denoted $\mathcal{T}_u$, on $\mathbb{R}$ such that the sets in $\mathscr{C}$ are clopen (i.e. open and closed).
(2) Show that $(\mathbb{R},\mathcal{T}_u)$ is first countable, separable, but not second countable.

Solution. (1) $\mathscr{C}$ is a base for a topology since $\mathbb{R} = \cup_{a<b\in\mathbb{R}}(a,b]$ and one can easily verify (via a case analysis) that $\mathscr{C}$ is stable under intersection. Since

$$(a,b]^c = (-\infty,a]\cup(b,\infty) = \cup_{n\in\mathbb{N}}(a-n,a]\cup\cup_{n\in\mathbb{N}}(b,b+n]$$

it follows that the elements in $\mathscr{C}$ are closed (and open by definition).

(2) Since by (1) every open set can be written as $\cup_{i\in I}(a_i, b_i]$ (wlog disjoint), one can verify that for all $x \in \mathbb{R}$, $\mathcal{B}_x \overset{\text{def}}{=} \{(x - \frac{1}{n}, x] \colon n \geqslant 1\}$ is a countable local base. It is not second countable. Indeed if $\mathcal{B}$ is a base, since $(-\infty, x]$ is open for every $x \in \mathbb{R}$, there be a open set $O_x \in \mathcal{B}$ with $sup(O_x) = x$ and thus $O_x \neq O_y$ whenever $x \neq y$. Therefore, any base $\mathcal{B}$ would be uncountable. It is immediate that the density of $\mathbb{Q}$ in $(\mathbb{R}, |\cdot|)$ implies its density in $(X, \mathcal{T}_u)$, and hence $(X, \mathcal{T}_u)$ is separable.

□

Problem 5. *Let (X, τ) be a topological space and let $acc(A)$ denote the set of accumulation points of A. Show that*

(1) $\overline{A} = A \cup acc(A)$

(2) $x \in acc(A)$ if and only if there exists a net in $A \setminus \{x\}$ that converges to x.

Solution. (1) Let $x \in acc(A)$ and assume that $x \in (\overline{A})^c = (A^c)^\circ$. Thus, there exists an open set $U \subset A^c$ such that $x \in U$ and $A \cap (U \setminus \{x\}) = \emptyset$, a contradiction. Therefore, $acc(A) \subset \overline{A}$ and $A \cup acc(A) \subset \overline{A}$.

We now show that $\overline{A} \subset A \cup acc(A)$. If $x \in \overline{A} \setminus A$ and assume that $x \notin acc(A)$. Then there is an open set U containing x and such that $A \cap (U \setminus \{x\}) = \emptyset$. But since $x \notin A$, $A \cap U = \emptyset$ and thus $A \subset U^c$, and in turn $\overline{A} \subset U^c$. Since $x \notin U^c$ we have $x \notin \overline{A}$; a contradiction.

(2) Let $x \in acc(A)$. Then for all $V \in \mathcal{N}_x$ we have $V \cap (A \setminus \{x\}) \neq \emptyset$. Pick $x_V \in A \cap V$ such that $x_v \neq x$. Consider the net $(x_v)_{V \in \mathcal{N}_x} \subset A$. IF $V \in \mathcal{N}_x$ and $U \subset V$ then $x \neq x_U \in U \subset V$, i.e. $(x_v)_{V \in \mathcal{N}_x}$ converges to x.

Assume now that there exists a net $\{x_\alpha\}_{\alpha \in D}$ in $A \setminus \{x\}$ that converges to x. Let $V \in \mathcal{N}_x$, then there exists $\beta \in D$ such that for all $\alpha \geqslant \beta$, $x \neq x_\alpha \in V$, and hence $(A \cap V) \setminus \{x\} \neq \emptyset$. Thus, $x \in acc(A)$.

□