MATH 220 Problems

1 Logical connectives and logical equivalence

Problem 1.1. *Show that* $P \vee P$ *is logically equivalent to* P*.*

Problem 1.2. *Show that* $P \wedge P$ *is logically equivalent to* P*.*

Problem 1.3. *Are the statement forms* $(P \wedge Q) \wedge R$ *and* $P \wedge (Q \wedge R)$ *logically equivalent?*

Problem 1.4. *Are the statement forms* $(P \vee Q) \vee R$ *and* $P \vee (Q \vee R)$ *logically equivalent?*

Problem 1.5. *Is the statement form* $(P \wedge Q) \vee ((\neg P) \wedge \neg Q)$ *a tautology, a contradiction, or neither?*

Problem 1.6. *Are the statement forms* $(P \vee Q) \wedge R$ *and* $P \vee (Q \wedge R)$ *logically equivalent?*

Problem 1.7. *Show that* $P \vee (Q \wedge R)$ *is logically equivalent to* $(P \vee Q) \wedge (P \vee R)$*.*

Problem 1.8. *Show that* $P \wedge (Q \vee R)$ *is logically equivalent to* $(P \wedge Q) \vee (P \wedge R)$*.*

Problem 1.9. *Are the statement forms* $P \implies (Q \vee R)$ *and* $(P \implies Q) \vee (P \implies R)$ *logically equivalent?*

Problem 1.10. *Are the statement forms* $P \implies (Q \wedge R)$ *and* $(P \implies Q) \wedge (P \implies R)$ *logically equivalent?*

Hint. *Try to use Problem 1.7.*

Problem 1.11. *Show that the statement forms* $(P \vee Q) \implies R$ *and* $(P \implies R) \wedge (Q \implies R)$ *are logically equivalent.*

Hint. *Try to use DeMorgan's Laws and Problem 1.7.*

Problem 1.12. *For all the statement forms below write a logically equivalent statement form that involves only the logical connective* $\neg$ *and* $\vee$*.*

1. $P \vee (Q \wedge R)$

2. $(P \vee Q) \wedge (P \vee R)$

3. $P \iff Q$

Hint. *Try to use DeMorgan's laws.*

Problem 1.13. *For all the statement forms below write a logically equivalent statement form that involves only the logical connective* $\neg$ *and* $\wedge$.

1. $P \wedge (Q \vee R)$

2. $(P \wedge Q) \vee (P \wedge R)$

3. $P \iff Q$

Hint. *Try to use DeMorgan's laws.*

Problem 1.14. *Are the statement forms* $[(\neg P) \implies [Q \wedge \neg Q]]$ *and* P *logically equivalent?*

2 Quantifiers

Problem 2.1. *What is the truth value of the statement* $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(\forall z \in \mathbb{R})[xy = xz]$*?*

Problem 2.2. *What is the truth value of the statement* $(\exists y \in \mathbb{R})(\forall x \in \mathbb{R})(\exists z \in \mathbb{R})[xy = xz]$*?*

Problem 2.3. *Let* $x_0 \in (a, b)$, $\ell \in \mathbb{R}$ *and* $f\colon (a, x_0) \cup (x_0, b) \to \mathbb{R}$. *We say that* ℓ *is the limit of* f *at* x_0, *and we write* $\lim_{x \to x_0} f(x) = \ell$, *if for all* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that if* x *satisfies* $0 < |x - x_0| < \delta$ *then* $|f(x) - \ell| < \varepsilon$. *Formally,*

$$\lim_{x \to x_0} f(x) = \ell \iff (\forall \varepsilon > 0)(\exists \delta > 0)(\forall x)[0 < |x - x_0| < \delta \implies |f(x) - \ell| < \varepsilon].$$

Negate the statement $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x)[0 < |x - x_0| < \delta \implies |f(x) - \ell| < \varepsilon]$.

Problem 2.4. *Give the definition of an even number using logical symbols and quantifiers.*

Problem 2.5. *Give the definition of a prime number using logical symbols and quantifiers.*

Problem 2.6. *Write a formal mathematical expression that expresses the fact that a given sequence* $(x_n)_{n \in \mathbb{N}}$ *does not have a real limit.*

Problem 2.7. *Negate the statement* $P : (\forall n \in \mathbb{Z})(\exists k \in \mathbb{Z})(n^2 + n + 1 = 2k)$. *Try to explain what* P *and* $\neg P$ *mean.*

Problem 2.8. *Let* f *be a function from* $\mathbb{R}$ *to* $\mathbb{R}$. *We say that* f *is strictly increasing if*

$$(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})[(x < y) \implies (f(x) < f(y))].$$

Negate the statement above.

Problem 2.9. *Let* f *be a function from* $\mathbb{R}$ *to* $\mathbb{R}$. *Define what it means for* f *to be strictly decreasing*

Problem 2.10. *Let* f *be a function from* $\mathbb{R}$ *to* $\mathbb{R}$. *Write a formal mathematical expression which expresses the fact that it is not true that* f *is strictly decreasing or strictly increasing.*

Problem 2.11. *Define formally what it means that an integer* k *divides an integer* n.

Problem 2.12. *Give a formal definition of what it means for a number* x *to be a rational number.*

Problem 2.13. *Give a formal definition of what it means for a number* x *to be a irrational number.*

3 Proofs

Problem 3.1. *Prove that the equation* (E): $7x - 2 = 0$ *has a unique solution in* $\mathbb{R}$.

Problem 3.2. *Prove that the equation* (E): $-3x + 8 = 0$ *has a unique solution in* $\mathbb{R}$.

Problem 3.3. *Let* $a, b, c \in \mathbb{R}$ *with* $a \neq 0$. *Prove that the equation* (E): $ax + b = c$ *has a unique solution in* $\mathbb{R}$.

Problem 3.4. *Let* a, b, *and* c *be integers. Prove that for all integers* m *and* n, *if* a *divides* b *and* a *divides* c, *then* a *divides* $(bm + cn)$.

Problem 3.5. *Prove that if* m *and* n *are even, then* $m + n$ *is even.*

Problem 3.6. *Prove that if* m *is even and* n *is odd, then* $m + n$ *is odd.*

Problem 3.7. *For all* $m, n \in \mathbb{Z}$, *if* m *is even, then* mn *is even.*

Problem 3.8. *Show that for all* $n \in \mathbb{Z}$, $4n + 7$ *is odd.*

Problem 3.9. *Let* n *be an integer. If* n^2 *is even, then* n *is even.*

Problem 3.10. *Let n be an integer. If n^3 is even, then n is even.*

Problem 3.11. *For this problem you can use the following fact that will be proven later: 3 does not divides n if and only if there exists an integer k and an integer $i \in \{1, 2\}$ such that $n = 3k + i$.*

Prove that for every integer n, if 3 divides n^2 then 3 divides n.

Problem 3.12. *Prove that there are no integers m and n such that $8m + 26n = 1$.*

Problem 3.13. *Are there integers m and n such that $m^2 = 4n + 3$?*

Problem 3.14. *Let $x \in \mathbb{R}$. If for all $\varepsilon > 0$, $|x| < 2\varepsilon$, then $x = 0$.*

Problem 3.15. *Prove that $\sqrt[3]{2}$ is irrational.*

Hint. Use Problem 3.10. □

Problem 3.16. *Show that $\sqrt{3}$ is irrational.*

Hint. Use Problem 3.11. □

Problem 3.17. *Show that $\log(3)$ is irrational.*

Hint. You can use the the following property of the log function: $x = \log(3) \iff 2^x = 3$ (no proof needed) You can also use the binomial formula (no proof needed). Everything else that you might need needs to be proven. □

Problem 3.18. *Prove that for all real numbers x and y with $y \geq 0$, if $x^2 \geq 4y$, then $x \geq 2\sqrt{y}$ or $x \leq -2\sqrt{y}$.*

Problem 3.19. *Prove that for all integer k, $k(k + 3)$ is even.*

Problem 3.20. *Prove that for all integer k, $(k + 1)(k + 6)$ is even.*

For the following problems we recall the definition of the absolute value function

$$|x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Problem 3.21. *Show that for all $x \in \mathbb{R}$, $|x| \geq 0$ with $|x| = 0$ if and only if $x = 0$.*

Problem 3.22. *Prove that for all real numbers x and y, $|x - y| = |y - x|$.*

Problem 3.23. *Prove that for all real numbers* x *and* y, $|xy| = |x||y|$.

Problem 3.24. *Let* $x \in \mathbb{R}$ *and* $M \geq 0$. *Show that* $|x| \leq M \iff -M \leq x \leq M$.

Problem 3.25. *Prove that for all real numbers* x *and* y, $|x+y| \leq |x| + |y|$.

Hint. You could use Problem 3.24. □

Problem 3.26. *Prove that for all* $x, y, z \in \mathbb{R}$, $|x-y| \leq |x-z| + |y-z|$.

Hint. You could use Problem 3.25. □

Problem 3.27. *Prove that for all real numbers* x *and* y, $\big||x| - |y|\big| \leq |x-y|$.

Hint. You could use Problem 3.25. □

Problem 3.28. *Let* x, y *be real numbers. Show that*

$$\forall \varepsilon > 0,\ x < y + \varepsilon \iff x \leq y.$$

Problem 3.29. *Let* x, y *be real numbers. Show that* $x > y - \varepsilon$ *for all* $\varepsilon > 0$ *if and only if* $x \geq y$.

Problem 3.30. *Prove that for all real numbers* x *and* y, *if* $x < y$, *then* $x < \frac{x+y}{2} < y$.

Problem 3.31. *Prove that for all positive real numbers* x, *the sum of* x *and its reciprocal is greater than* 2.

Problem 3.32.

1. *Prove that for all* $x, y \in \mathbb{R}^+$, $\sqrt{xy} \leq \frac{x+y}{2}$.
2. *Show that that for all* $x, y \in \mathbb{R}^+$, $\sqrt{xy} = \frac{x+y}{2}$ *if and only if* $x = y$

4 Applications of the Principle of Mathematical Induction

Problem 4.1. *Prove that for all integers* $n \geq 1$,

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Problem 4.2. *Prove that for all integers* $n \geq 0$,

$$\sum_{k=0}^{n} 2^k = 2^{n+1} - 1.$$

Problem 4.3. *Prove that for all integers $n \geq 1$,*

$$\sum_{k=1}^{n}(2k-1) = n^2.$$

Problem 4.4. *Prove that for all integers $n \geq 1$,*

$$\sum_{k=1}^{n}\frac{1}{k(k+1)} = \frac{n}{n+1}.$$

Problem 4.5. *Prove that for all integers $n \geq 1$,*

$$\sum_{k+1}^{n}(2k-1)^2 = \frac{4n^3-n}{3}.$$

Problem 4.6. *Conjecture a formula for $\sum_{k=1}^{n}(-1)^k k^2$, for all $n \geq 1$ and then prove the formula is correct using induction.*

Problem 4.7. *Prove that for all integers $n \geq 1$, $n < 10^n$.*

Problem 4.8. *Prove that for all integers $n \geq 7$, $\left(\frac{4}{3}\right)^n > n$.*

Problem 4.9. *Prove that for all integers $n \geq 1$, $n^3 + 8n + 9$ is divisible by 3.*

Problem 4.10. *Prove that for all integers $n \geq 1$, $3^{2n} - 1$ is divisible by 8.*

Problem 4.11. *Prove that for all integers $n \geq 5$, $n^2 < 2^n$.*

Problem 4.12. *Prove that for all integers $n \geq 4$, $2^n < n!$.*

Problem 4.13. *Assuming that $(1+\frac{1}{n})^n < e$, for all $n \geq 1$, prove that for all $n \geq 1$, $n! > (\frac{n}{e})^n$.*

Problem 4.14. *Prove that for all positive integers n, $4^n - 1$ is divisible by 3.*

Problem 4.15. *Let $a_1 = 2$, and let $a_{n+1} = \frac{1}{2}(a_n + 3)$ for all $n \geq 1$.*

(a) Prove that for all positive integers n, $a_n < a_{n+1}$.

(b) Prove that for all positive integers n, $a_n < 3$.

(c) Prove that for all positive integers n, $a_n = 3 - \frac{1}{2^{n-1}}$.

Problem 4.16. *Let $r \in \mathbb{R}$ with $r \neq 1$. Prove that*

$$\sum_{k=0}^{n-1} r^k = \frac{1-r^n}{1-r}.$$

Problem 4.17. *Prove Bernoulli's Inequality: Let $x > -1$. Then for all $n \in \mathbb{N}$, $(1+x)^n \geq 1 + nx$.*

Problem 4.18. *Let $x, y \in \mathbb{R}$. Prove the binomial theorem: for all integers $n \geq 1$,*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

Problem 4.19. *Let n be an integer. Show that if n is even then n^k is even for all $k \in \mathbb{N}$.*

5 Applications of the Principle of Strong Mathematical Induction

Problem 5.1. *For $i \in \mathbb{N}$, let p_i denote the ith prime number, so that*

$$p_1 = 2, \qquad p_2 = 3, \qquad p_3 = 5, \ldots.$$

Prove that for all $n \in \mathbb{N}$, $p_n \leq 2^{2^{n-1}}$.

Hint. *For the induction step, given $m \in \mathbb{N}$, show that $p_{m+1} \leq p_1 p_2 \cdots p_m + 1$.*

Problem 5.2. *Show that the principle of strong mathematical induction implies the principle of mathematical induction.*

Problem 5.3. *Show that the principle of mathematical induction implies the principle of strong mathematical induction.*

6 Sequences defined by a recurrence relation

Problem 6.1. *Let $a_1 = 2$, $a_2 = 4$, and $a_{n+1} = 7a_n - 10a_{n-1}$ for all $n \geq 2$. Conjecture a closed formula for a_n and prove your result.*

Problem 6.2. *Let $a_1 = 3$, $a_2 = 4$, and $a_{n+1} = \frac{1}{3}(2a_n + a_{n-1})$ for all $n \geq 2$. Prove that for all positive integers n, $3 \leq a_n \leq 4$.*

Problem 6.3. *Consider the sequence $(a_n)_{n=1}^{\infty}$ recursively defined as $a_1 = 1$, $a_2 = 8$ and for all $n \geq 3$, $a_n = a_{n-1} + 2a_{n-2}$. Show that for all $n \geq 1$, $a_n = 3 \cdot 2^{n-1} + 2(-1)^n$.*

Problem 6.4. *Consider the sequence $(a_n)_{n=1}^{\infty}$ recursively defined as $a_1 = 2$, $a_2 = 4$ and for all $n \geq 3$, $a_n = 3a_{n-1} - 2a_{n-2}$. For all $n \geq 1$, find a closed formula for a_n.*

7 Set Theory

7.1 Subsets

Problem 7.1. *Prove that* $X \subseteq Y$ *where* $X = \{n \in \mathbb{Z} \mid n \text{ is a multiple of } 6\}$ *and* $Y = \{n \in \mathbb{Z} \mid n \text{ is even}\}$.

Problem 7.2. *Consider the sets*

$$A = \{n \in \mathbb{Z} \mid (\exists k \in \mathbb{Z})(n = 12k + 11)\},$$
$$B = \{n \in \mathbb{Z} \mid (\exists j \in \mathbb{Z})(n = 4j + 3)\}.$$

(a) Is $A \subseteq B$*? Prove or disprove.*

(b) Is $B \subseteq A$*? Prove or disprove.*

Problem 7.3. *Consider the sets*

$$A = \{n \in \mathbb{Z} \mid (\exists k \in \mathbb{Z})(n = 4k + 1)\},$$
$$B = \{n \in \mathbb{Z} \mid (\exists j \in \mathbb{Z})(n = 4j - 7\}.$$

Prove that $A = B$.

Problem 7.4. *Consider the sets*

$$A = \{n \in \mathbb{Z} \mid (\exists k \in \mathbb{Z})(n = 3k)\},$$
$$B = \{n \in \mathbb{Z} \mid (\exists i, j \in \mathbb{Z})(n = 15i + 12j)\}.$$

Prove that $A = B$.

Problem 7.5. *Prove that* $X = \{n \in \mathbb{Z} \mid n + 5 \text{is odd}\}$ *is the set of all even integers.*

7.2 Complements

Problem 7.6. *Let* A *and* B *be subsets of an ambient set* U*. Prove that* $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$.

7.3 Arbitrary unions and intersections

Problem 7.7. *For* $i \in \mathbb{N}$*, let* $A_i = (-i, i)$*. Compute* $\bigcup_{i=1}^{\infty} A_i$.

Problem 7.8. *For $i \in \mathbb{N}$, let $A_i = (-i, i)$. Compute $\bigcap_{i=1}^{\infty} A_i$.*

Problem 7.9. *For $i \in \mathbb{N}$, let $A_i = \left[0, 1 - \frac{1}{i}\right]$. Compute $\bigcup_{i \in \mathbb{N}} A_i$.*

Problem 7.10. *For $i \in \mathbb{N}$, let $A_i = \left[0, 1 - \frac{1}{i}\right]$. Compute $\bigcap_{i \in \mathbb{N}} A_i$.*

Problem 7.11. *Let $X_n = (\frac{2}{n}, 2n]$ for every integer $n \geq 1$.*

1. *Compute $\bigcup_{n=1}^{\infty} X_n$.*

2. *Compute $\bigcap_{n=2}^{\infty} X_n$.*

Problem 7.12. *Let I be a nonempty set and let $\{A_i : i \in I\}$ be an indexed family of sets. Let X be a non-empty set. Suppose that for all $i \in I$, $X \subseteq A_i$. Prove that $X \subseteq \bigcap_{i \in I} A_i$.*

Problem 7.13. *Let $\{A_i : i \in \mathbb{N}\}$ be an indexed family of sets. Assume that for all $i \in \mathbb{N}$, $A_{i+1} \subseteq A_i$. Prove that $\bigcup_{i \in \mathbb{N}} A_i = A_1$.*

Problem 7.14. *Let $(X_i)_{i \in I}$ be a collection of subsets of an ambient set U. Show that*

$$\overline{\bigcap_{i \in I} X_i} = \bigcup_{i \in I} \overline{X}_i.$$

Problem 7.15. *Let $(X_i)_{i \in I}$ be a collection of subsets of an ambient set U. Show that*

$$\overline{\bigcup_{i \in I} X_i} = \bigcap_{i \in I} \overline{X}_i.$$

7.4 More problems

Problem 7.16. *Let $A = \{x + y\sqrt{2} \mid x, y \in Q\} \subseteq \mathbb{R}$.*

(a) Prove that for all $x, y \in Q$, $x + y\sqrt{2} = 0$ if and only if $x = y = 0$.

(b) Prove that for all $z_1, z_2 \in A$, $z_1 + z_2, z_1 z_2 \in A$ and , for $z_2 \neq 0$, $\frac{z_1}{z_2} \in A$.

Problem 7.17. *We say that the sequence of sets $(X_n)_{n=1}^{\infty}$ is increasing, or an ascending chain, if $X_1 \subseteq X_2 \subseteq X_3 \subseteq \cdots \subseteq X_n \subseteq X_{n+1} \subseteq \ldots$. Formally, $(X_n)_{n=1}^{\infty}$ is increasing if*

$$(\forall n \in \mathbb{N})[X_n \subseteq X_{n+1}].$$

Show that the sequence of sets $(X_n)_{n=1}^{\infty}$ is increasing if and only if

$$(\forall n \in \mathbb{N})(\forall k \in \mathbb{N})[(n \leq k) \implies (X_n \subseteq X_k)].$$

Problem 7.18. *We say that the sequence of sets* $(X_n)_{n=1}^{\infty}$ *is decreasing, or a descending chain, if* $X_1 \supseteq X_2 \supseteq X_3 \supseteq \cdots \supseteq X_n \supseteq X_{n+1} \supseteq \ldots$. *Formally,* $(X_n)_{n=1}^{\infty}$ *is increasing if*

$$(\forall n \in \mathbb{N})[X_n \subseteq X_{n+1}].$$

Show that the sequence of sets $(X_n)_{n=1}^{\infty}$ *is decreasing if and only if for all* $n, k \in \mathbb{N}$ *if* $n \leq k$ *then* $X_n \supseteq X_k$.

Problem 7.19. *Let* X *and* Y *be subsets of a universal set* U. *Show that* $\overline{X \cap Y} = \overline{X} \cup \overline{Y}$.

8 Functions

8.1 Composition

Problem 8.1. *Let* $f, g : \mathbb{R} \to \mathbb{R}$ *be defined for all* $x \in \mathbb{R}$ *as* $f(x) = x^2 - 3x$ *and* $g(x) = 5x - 2$.

1. *Is it possible to define* $f \circ g$? *If it is, what is* $f \circ g$.

2. *Is it possible to define* $g \circ f$? *If it is, what is* $g \circ f$.

3. *Are* $f \circ g$ *and* $g \circ f$ *equal? (Justify your answer)*

Problem 8.2. *Let* $f, g : \mathbb{Z} \to \mathbb{Z}$ *be defined for all* $n \in \mathbb{Z}$ *as* $f(n) = 2n + 3$ *and*

$$g(n) = \begin{cases} 2n - 1 & \text{if } n \text{ is even,} \\ n + 1 & \text{if } n \text{ is odd.} \end{cases}$$

1. *Is it possible to define* $f \circ g$? *If it is, what is* $f \circ g$.

2. *Is it possible to define* $g \circ f$? *If it is, what is* $g \circ f$.

3. *Are* $f \circ g$ *and* $g \circ f$ *equal? (Justify your answer)*

8.2 Injectivity, surjectivity, bijectivity

Problem 8.3. *For* $f : \mathbb{R} \to \mathbb{R}$ *defined by* $f(x) = x + |x|$, *determine if:*

1. f *is injective,*

2. f *is surjective,*

3. f *is bijective.*

8.3 Composition and injectivity/surjectivity

Problem 8.4. *Let W, X, Y be nonempty sets. Let $f : W \to X$, $g : X \to Y$ be functions. Show that if $g \circ f$ is surjective, then g is surjective.*

Problem 8.5. *Let W, X, Y be nonempty sets. Let $f : W \to X$, $g : X \to Y$ be functions. Show that if $g \circ f$ is injective, then f is injective.*

Problem 8.6. *Let X and Y be nonempty sets and let $f : X \to Y$ be a function. Prove that f is injective if and only if for all sets Z, for all functions $h : Z \to X$ and $k : Z \to X$, if $f \circ h = f \circ k$, then $h = k$.*

Problem 8.7. *Let X and Y be nonempty sets and let $f : X \to Y$ be a function. Prove that f is surjective if and only if for all sets Z, for all functions $h : Y \to Z$ and $k : Y \to Z$, if $h \circ f = k \circ f$, then $h = k$.*

9 Injectivity/surjectivity and invertibility

Problem 9.1. *Let X and Y be nonempty sets and $f\colon X \to Y$ be a function. Prove that f is injective if and only if f is left-invertible.*

Problem 9.2. *Let X and Y be nonempty sets, and $f\colon X \to Y$ be a function. Suppose that f has a right-inverse h. Prove that f is surjective.*

10 Functions and sets

Problem 10.1. *Let X and Y be nonempty sets, and $f\colon X \to Y$ be an injective function. Let A be a subset of X. Prove that $f^{-1}(f(A)) = A$.*

Problem 10.2. *Let X and Y be nonempty sets, and $f\colon X \to Y$ be an surjective function. Let A be a subset of Y. Prove that $f(f^{-1}(A)) = A$.*

11 Supplementary problems

Problem 11.1. *Let $f_1\colon X_1 \to X_2$, $f_2\colon X_2 \to X_3$, $f_3\colon X_3 \to X_4$ and $f_4\colon X_4 \to X_5$. Show that $((f_4 \circ f_3) \circ f_2) \circ f_1 = f_4 \circ (f_3 \circ (f_2 \circ f_1))$.*

Problem 11.2. *Let X and Y be nonempty sets, and $f\colon X \to Y$ be a function. Prove that f is surjective then f is right-invertible.*

Problem 11.3. *Let $f_1\colon X_1 \to X_2$, $f_2\colon X_2 \to X_3$, $f_3\colon X_3 \to X_4$ be three injective functions. Show that $f_3 \circ f_2 \circ f_1$ is injective.*