REAL ANALYSIS MATH 608
HOMEWORK #2

Problem 1. *Let X and Y be topological spaces and $f\colon X \to Y$ be a map. Show that f is continuous at $x \in X$ if and only if for every net $(x_\alpha)_{\alpha\in D}$ converging to x, $(f(x_\alpha))_{\alpha\in D}$ converges to $f(x)$.*

Problem 2. *Let $(x_\alpha)_{\alpha\in D}$ be a net in a topological space X and $x \in X$. Show that the following assertions are equivalent:*

(1) $(x_\alpha)_{\alpha\in D}$ converges to x.
(2) Every subnet of $(x_\alpha)_{\alpha\in D}$ has a cofinal subnet that converges to x.
(3) Every subnet of $(x_\alpha)_{\alpha\in D}$ has a subnet that converges to x.
(4) Every cofinal subnet of $(x_\alpha)_{\alpha\in D}$ has a subnet that converges to x.
(5) Every cofinal subnet of $(x_\alpha)_{\alpha\in D}$ has a cofinal subnet that converges to x.

Hint: There is essentially only one implication that requires a proof; which one?

Problem 3. *Let X be a topological space and denote by $B(X)$ (resp. $BC(X)$) the space of (real or complex valued) bounded (resp. bounded and continuous) functions on X. These spaces are equipped with their natural vector space structure and the metric $d_u(f,g) \overset{\text{def}}{=} \sup_{x\in X}|f(x)-g(x)|$.*

(1) Show that d_u is a complete metric on $B(X)$.
(2) Show that $BC(X)$ is a closed subspace of $B(X)$ (with respect to the topology induced by the uniform metric).
(3) What can you deduce from (2) regarding the completness of $BC(X)$?

Problem 4. *Let $(x_\alpha)_{\alpha\in D}$ be a net in a topological space X, and for each $\alpha \in (D,\leqslant)$ let $T_\alpha \overset{\text{def}}{=} \{x_\beta\colon \beta \geqslant \alpha\}$ Show that the set of cluster points of $(x_\alpha)_{\alpha\in D}$ is $\bigcap_{\alpha\in A} \overline{T_\alpha}$.*

Problem 5. *A filter on a set X is a collection $\mathcal{F}$ of non-empty subsets of X that is stable under taking supersets and finite intersections. A filter in a topological space is said to converge to a point if it contains the neighborhood system of the point.*

(1) Verify that the neighborhood system of a point in a topological space is a filter.
(2) Show that a topological space X is Hausdorff if and only every convergent filter in X has a unique limit.
(3) Given a net $(x_\alpha)_{\alpha\in D}$ in a topological space X, construct a filter $\mathcal{F}$ on X such that, $(x_\alpha)_{\alpha\in D}$ converges to $x \in X$ if and only if $\mathcal{F}$ converges to $x \in X$.
(4) Given a filter $\mathcal{F}$ on a topological space X, construct a net $(x_\alpha)_{\alpha\in D}$ in X such that, $(x_\alpha)_{\alpha\in D}$ converges to $x \in X$ if and only if $\mathcal{F}$ converges to $x \in X$.

Hint: For (3), think of the tails of the net. For (4), direct the set $\{(x,F)\colon x \in F \in \mathcal{F}\}$ in an adequate way.