REAL ANALYSIS MATH 608
HOMEWORK #2

Problem 1. *Let X and Y be topological spaces and $f\colon X \to Y$ be a map. Show that f is continuous at $x \in X$ if and only if for every net $(x_\alpha)_{\alpha\in D}$ converging to x, $(f(x_\alpha))_{\alpha\in D}$ converges to $f(x)$.*

Solution. Assume that f is continuous at $x \in X$ and let $(x_\alpha)_{\alpha\in D}$ be a net converging to x. Let $U \in \mathcal{N}_{f(x)}$, then by continuity $f^{-1}(U) \in \mathcal{N}_x$ and hence $(x_\alpha)_{\alpha\in D}$ is eventually in $f^{-1}(U)$, which in turn implies that $(f(x_\alpha))_{\alpha\in D}$ is eventually in U.

Now, if f is not continuous at x, then there is $U \in \mathcal{N}_{f(x)}$ such that $f^{-1}(U) \notin \mathcal{N}_x$. This implies that x does not belong to the interior of $f^{-1}(U)$, and hence x is in the closure of the complement of $f^{-1}(U)$. Therefore, there is a net $(x_\alpha)_{\alpha\in D}$ in $f^{-1}(U)^c = f^{-1}(U^c)$ that converges to x, and the net $(f(x_\alpha))_{\alpha\in D}$ cannot converge to $f(x)$ since it is never in U. □

Problem 2. *Let $(x_\alpha)_{\alpha\in D}$ be a net in a topological space X and $x \in X$. Show that the following assertions are equivalent:*

(1) $(x_\alpha)_{\alpha\in D}$ converges to x.
(2) Every subnet of $(x_\alpha)_{\alpha\in D}$ has a cofinal subnet that converges to x.
(3) Every subnet of $(x_\alpha)_{\alpha\in D}$ has a subnet that converges to x.
(4) Every cofinal subnet of $(x_\alpha)_{\alpha\in D}$ has a subnet that converges to x.
(5) Every cofinal subnet of $(x_\alpha)_{\alpha\in D}$ has a cofinal subnet that converges to x.

Hint: There is essentially only one implication that requires a proof; which one?

Solution. (1) $\implies$ (2) $\implies$ (3) $\implies$ (4) is trivial, as well as (5) $\implies$ (4) and (1) $\implies$ (5). It remains to prove (4) $\implies$ (1). Assume that $(x_\alpha)_{\alpha\in D}$ does not converge to x. Then there is $V \in \mathcal{N}_x$ such that $(x_\alpha)_{\alpha\in D}$ is frequently in V^c, and hence there is a cofinal subnet which is eventually in V^c, and thus the same is true for all its subnets which certainly cannot converge to x. □

Problem 3. *Let X be a topological space and denote by $B(X)$ (resp. $C_b(X)$) the space of (real or complex valued) bounded (resp. bounded and continuous) functions on X. These spaces are equipped with their natural vector space structure and the metric $d_u(f,g) \overset{\text{def}}{=} \sup_{x\in X}|f(x)-g(x)|$.*

(1) Show that d_u is a complete metric on $B(X)$.
(2) Show that $C_b(X)$ is a closed subspace of $B(X)$ (with respect to the topology induced by the uniform metric).
(3) What can you deduce from (2) regarding the completness of $C_b(X)$?

Solution. (1) That d_u is a metric is elementary (symmetry and definitness is clear, and the triangle inequality is a consequence of the triangle inequality for the module). As for completness, let $(f_n)_n$ be a Cauchy sequence in $B(X)$ with respect to d_u. Then, for all $x \in X$, $(f_n(x))_n$ is a Cauchy sequence of real or complex numbers which is convergent by completness of the fields equipped with the relevant module. Let $f(x) \overset{\text{def}}{=} \lim_n f_n(x)$. Then $\lim_n d_u(f_n, f) = 0$. Indeed, for $\varepsilon > 0$ let $N \in \mathbb{N}$ such that $n, k \geqslant N \implies d_u(f_n, f_k) \leqslant \varepsilon$. Then, for all $x \in X$, and $n, k \geqslant N$, $|f_n(x) - f_k(x)| \leqslant \varepsilon$, Taking the limit when $n \to \infty$ we have $|f(x) - f_k(x)| \leqslant \varepsilon$ for all $x \in X$ and all $k \geqslant N$, i.e. for all $k \geqslant N$, $d_u(f, f_k) \leqslant \varepsilon$; which proves the claim. f is bounded as it is the uniform limit of bounded functions. Indeed, let $N \geqslant 1$ such that for all $k \geqslant N$, $d_u(f, f_k) \leqslant 1$, then $\sup_{x \in X} |f(x)| = d_u(f, 0) \leqslant d_u(f, f_N) + d_u(f_N, 0) \leqslant 1 + \sup_{x \in X} |f_N(x)| < \infty$.

(2) We have already seen above that the uniform limit of a sequence of bounded functions is bounded. But a uniform limit of a sequence of continuous function is also continuous. Indeed, let $(f_n)_n$ be a sequence of continuous functions that converges uniformly to f, and for $\varepsilon > 0$ let $N \geqslant 1$ such that $d_u(f, f_N) < \varepsilon$. For $x_0 \in X$, let $U \in \mathcal{N}_{x_0}$ such that for all $x \in U$, $|f_N(x) - f_N(x_0)| \leqslant \varepsilon$. Then, for all $x \in U$ $|f(x) - f(x_0)| \leqslant |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| \leqslant 3\varepsilon$, and the conclusion follows.

(3) Since a closed subset of a complete metric space is complete (a Cauchy sequence in closed subset converges to a point (by completness) that remains in the set (by closedness)), it follows from (2) that $C_b(X)$ equipped with the uniform metric is complete.

□

Problem 4. *Let $(x_\alpha)_{\alpha \in D}$ be a net in a topological space X, and for each $\alpha \in (D, \leqslant)$ let $T_\alpha \overset{\text{def}}{=} \{x_\beta \colon \beta \geqslant \alpha\}$ Show that the set of cluster points of $(x_\alpha)_{\alpha \in D}$ is $\bigcap_{\alpha \in D} \overline{T_\alpha}$.*

Solution. Let $z \in \bigcap_{\alpha \in D} \overline{T_\alpha}$, then for all $\alpha \in D$, $z \in \overline{T_\alpha}$ and for all $V \in \mathcal{N}_z$, $V \cap T_\alpha \neq \emptyset$. Therefore, for all $\alpha \in D$, there is $\beta \geqslant \alpha$ such that $x_\beta \in V$, i.e. $(x_\alpha)_\alpha$ is frequently in V, and thus z is a cluster point of $(x_\alpha)_\alpha$.

Now, if x is a cluster point of $(x_\alpha)_\alpha$, then for all $V \in \mathcal{N}_x$, all $\alpha \in D$, there is $\beta \geqslant \alpha$ such that $x_\beta \in V$; i.e. for all $\alpha \in D$, $x \in \overline{T_\alpha}$ and every cluster point belongs to $\bigcap_{\alpha \in D} \overline{T_\alpha}$, which concludes the proof.

□

Problem 5. *A filter on a set X is a collection $\mathcal{F}$ of non-empty subsets of X that is stable under taking supersets and finite intersections. A filter in a topological space is said to converge to a point if it contains the neighborhood system of the point.*

(1) Verify that the neighborhood system of a point in a topological space is a filter.
(2) Show that a topological space X is Hausdorff if and only every convergent filter in X has a unique limit.
(3) Given a net $(x_\alpha)_{\alpha \in D}$ in a topological space X, construct a filter $\mathcal{F}$ on X such that, $(x_\alpha)_{\alpha \in D}$ converges to $x \in X$ if and only if $\mathcal{F}$ converges to $x \in X$.
(4) Given a filter $\mathcal{F}$ on a topological space X, construct a net $(x_\alpha)_{\alpha \in D}$ in X such that, $(x_\alpha)_{\alpha \in D}$ converges to $x \in X$ if and only if $\mathcal{F}$ converges to $x \in X$.

Hint: For (3), think of the tails of the net. For (4), direct the set $\{(x, F) \colon x \in F \in \mathcal{F}\}$ in an adequate way.

Solution. (1) This is true since the intersection of two open sets is an open set.

(2) Assume that $x \neq y$, X is Hausdorff and that there is a filter $\mathcal{F}$ that converges to x and to y. Let $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$. Since U and V are in $\mathcal{F}$, so is their intersection which therefore is non empty, a contradiction. Now if X is not Hausdorff, let $x \neq y$ be such that for all neighborhoods V of x and W of y one has $V \cap W \neq \emptyset$. Consider the filter $\mathcal{F} = \{Z \subset X \colon Z \supset V \cap W \text{ for some } V \in \mathcal{N}_x, W \in \mathcal{N}_y\}$. Then $\mathcal{F}$ converges to x and to y.

(3) Define $\mathcal{F} = \{Z \subset X \colon \exists \alpha \in D \text{ s.t. } Z \supset T_\alpha\}$. $\mathcal{F}$ is a filter because D is directed, and the equivalence follows immediately from the definition of $\mathcal{F}$.

(4) Direct the set $D = \{(x, F) \colon x \in F \in \mathcal{F}\}$ as follows $(x, F) \preceq (y, G) \iff G \subset F$, and define $z_{(x,F)} = x$ for all $(x, F) \in D$. If $\mathcal{F}$ converges to x, then given $V \in \mathcal{N}_x$, $V \in \mathcal{F}$ and hence $(x, V) \in D$. For all $(t, W) \in D$ such that $(x, V) \preceq (t, W)$ one has $z_{(t,W)} = t \in W \subset V$, which means that $(z_\alpha)_{\alpha \in D}$ is eventually in V and

hence converges to x. Assume that $(z_\alpha)_{\alpha\in D}$ converges to x. Then given $V \in \mathcal{N}_x$ there is $(t,F) \in D$ such that for all $(t,F) \leq (s,G)$, $z_{(s,G)} \in V$. Since for all $g \in G$, it is plain that $(t,F) \leq (g,F)$, then $z_{(g,F)} = g \in V$, and hence $F \subset V$. Therefore $V \in \mathcal{F}$ and $\mathcal{F}$ converges to x.

□