MATH 300
Practice EXAM II
Duration: 120 minutes

LAST NAME: ______________________________FIRST NAME: ______________________________

SECTION NUMBER: _______________

DIRECTIONS:

1. The use of a calculator, laptop, computer, book, or lecture notes is prohibited.
2. *Show all your work neatly and concisely.*

THE AGGIE CODE OF HONOR
"An Aggie does not lie, cheat, or steal, or tolerate those who do."

Signature: ____

Problem	Points Awarded	Points
1		20
2		10
3		12
4		12
5		12
6		10
7		12
8		12
TOTAL		100

Problem 1 *(2 points each question)* ***CIRCLE the correct answer.***
Let X and Y be sets. We say that X is a subset of Y, and write $X \subseteq Y$, if

1. $[x \in X \implies x \in Y]$
2. $(\exists x)[x \in X \implies x \in Y]$
3. $(\forall x)[x \in X \implies x \in Y]$
4. *none of the above.*

Let X be a set. Then $\emptyset \subseteq X$.

1. *TRUE*
2. *FALSE*

Let X, Y, and Z be sets. If $X \subseteq Y$ and $Y \subseteq Z$, then

1. $Z \subseteq X$.
2. $X \subseteq Z$.
3. *none of the above.*

We say that two sets X and Y are equal, written $X = Y$, if

1. $(X \subseteq Y) \vee (Y \subseteq X)$.
2. $(X \subseteq Y) \wedge (Y \subseteq X)$.
3. *none of the above.*

Let X be a set. Then,

1. $X \cup \emptyset = X$
2. $X \cup \emptyset = \emptyset$.
3. *none of the above.*

Let X be a set. Then,

1. $X \cap \emptyset = X$
2. $X \cap \emptyset = \emptyset$.
3. *none of the above.*

Let X and Y be sets. Then $X \cap Y \subseteq X$.

1. *TRUE*
2. *FALSE*

Let X and Y be subsets of a universal set U. Which of the following statements is FALSE.

1. $\overline{U} = \emptyset$,
2. $\overline{\emptyset} = U$.
3. $X - Y = Y \cap \overline{X}$,
4. $\emptyset - X = \emptyset$,
5. $X - \emptyset = X$.
6. $\overline{\overline{X}} = X$
7. *none of the above.*

For all integers k, $k(k+3)$ is even.

1. *TRUE*
2. *FALSE*

For the following problem we recall the definition of the absolute value function

$$|x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Let $x, y \in \mathbb{R}$. Which of the following statements is FALSE.

1. $|xy| = |yx|$
2. $|x - y| = |y - x|$
3. $|x + y| \leq |x| + |y|$
4. $|x + y| > |x| + |y|$.
5. *none of the above.*

Problem 2 *(10 points) Prove that for all integers $n \geq 1$,*

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}.$$

Problem 3 *(12 points) Let* $a_1 = 2$, $a_2 = 4$, *and* $a_{n+1} = 7a_n - 10a_{n-1}$ *for all* $n \geq 2$. *Conjecture a closed formula for* a_n *and prove your result.*

Problem 4 *(12 points) Prove that for all integers* $n \geq 1$, $3^{2n} - 1$ *is divisible by* 8.

Problem 5 *(12 points)*

1. *(7 points) Prove that for all integers* $n \geq 1$,

$$\sum_{k=0}^{n-1} 2^k = 2^n - 1.$$

For the next two questions, for $i \in \mathbb{N}$, let p_i denote the ith prime number, so that

$$p_1 = 2, \qquad p_2 = 3, \qquad p_3 = 5, \ldots.$$

2. *(2 points) Given an integer $m \geq 1$, show that $p_{m+1} \leq p_1 p_2 \cdots p_m + 1$.*
   (Hint: you could show by contradiction that if $1 \leq i \leq m$ then p_i does not divide $p_1 p_2 \cdots p_m + 1$.)

3. *(3 points) Prove that for all integers $n \geq 1$, $p_n \leq 2^{2^{n-1}}$.*
   (Hint: You can use question 1. and question 2.)

Problem 6 *(10 points) Consider the sets*

$$A = \{n \in \mathbb{Z} \mid (\exists k \in \mathbb{Z})(n = 4k + 1)\},$$
$$B = \{n \in \mathbb{Z} \mid (\exists j \in \mathbb{Z})(n = 4j - 7\}.$$

Prove that $A = B$.

Problem 7 *(12 points) Let X and Y be subsets of a universal set U. Show that* $\overline{X \cap Y} = \overline{X} \cup \overline{Y}$.

Problem 8 *(12 points) For this problem you can use the Archimedean principle.*

Archimedean Principle: *For all real number $x > 0$ there exists an integer $n \geq 1$ such that $x < n$.*

Let $X_n = (\frac{2}{n}, 7n]$ for $n \geq 1$.

1. *(7 points) Compute $\bigcup_{n=1}^{\infty} X_n$.*

2. (5 points) Compute $\bigcap_{n=1}^{\infty} X_n$.