MATH 300 Problems with solutions

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Contents

1 Logical connectives and equivalences, Boolean Calculus

Problem 1.1. *Show that $P \vee P$ is logically equivalent to P.*

Solution of Problem 1.1. We write the truth table for $P \vee P$. □

P	P	$P \vee P$
T	T	T
F	F	F

Problem 1.2. *Show that $P \wedge P$ is logically equivalent to P.*

Solution of Problem 1.2. We write the truth table for $P \wedge P$. □

P	P	$P \wedge P$
T	T	T
F	F	F

Problem 1.3. *Are the statement forms $(P \wedge Q) \wedge R$ and $P \wedge (Q \wedge R)$ logically equivalent?*

Solution of Problem 1.3. We write a truth table for $(P \wedge Q) \wedge R$ and $P \wedge (Q \wedge R)$:

P	Q	R	$P \wedge Q$	$Q \wedge R$	$(P \wedge Q) \wedge R$	$P \wedge (Q \wedge R)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	F	F	F
T	F	F	F	F	F	F
F	T	T	F	T	F	F
F	T	F	F	F	F	F
F	F	T	F	F	F	F
F	F	F	F	F	F	F

Since the columns corresponding to $(P \wedge Q) \wedge R$ and $P \wedge (Q \wedge R)$ are identical, the two statement forms are equivalent. □

Problem 1.4. *Are the statement forms $(P \vee Q) \vee R$ and $P \vee (Q \vee R)$ logically equivalent?*

Solution of Problem 1.4. We write a truth table for $(P \vee Q) \vee R$ and $P \vee (Q \vee R)$:

P	Q	R	$P \vee Q$	$Q \vee R$	$(P \vee Q) \vee R$	$P \vee (Q \vee R)$
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	T	T	T	T
T	F	F	T	F	T	T
F	T	T	T	T	T	T
F	T	F	T	T	T	T
F	F	T	F	T	T	T
F	F	F	F	F	F	F

Since the columns corresponding to $(P \vee Q) \vee R$ and $P \vee (Q \vee R)$ are identical, the two statement forms are equivalent. □

Problem 1.5. *Is the statement form $(P \wedge Q) \vee ((\neg P) \wedge \neg Q)$ a tautology, a contradiction, or neither?*

Solution of Problem 1.5. We write a truth table for $(P \wedge Q) \vee ((\neg P) \wedge \neg Q)$:

P	Q	$P \wedge Q$	$(\neg P) \wedge \neg Q$	$(P \wedge Q) \vee ((\neg P) \wedge \neg Q)$
T	T	T	F	T
T	F	F	F	F
F	T	F	F	F
F	F	F	T	T

Since $(P \wedge Q) \vee ((\neg P) \wedge \neg Q)$ is neither always true nor always false, the statement form is neither a tautology nor a contradiction. □

Problem 1.6. *Are the statement forms $(P \vee Q) \wedge R$ and $P \vee (Q \wedge R)$ logically equivalent?*

Solution of Problem 1.6. We write a truth table for $(P \vee Q) \wedge R$ and $P \vee (Q \wedge R)$:

P	Q	R	$P \vee Q$	$Q \wedge R$	$(P \vee Q) \wedge R$	$P \vee (Q \wedge R)$
T	T	T	T	T	T	T
T	T	F	T	F	F	T
T	F	T	T	F	T	T
T	F	F	T	F	F	T
F	T	T	T	T	T	T
F	T	F	T	F	F	F
F	F	T	F	F	F	F
F	F	F	F	F	F	F

Since the columns corresponding to $(P \vee Q) \wedge R$ and $P \vee (Q \wedge R)$ differ (in the second and fourth rows), the two statements are not logically equivalent. □

Problem 1.7. *Show that $P \vee (Q \wedge R)$ is logically equivalent to $(P \vee Q) \wedge (P \vee R)$.*

Solution of Problem 1.7. We write a truth table for $P \vee (Q \wedge R)$ and $(P \vee Q) \wedge (P \vee R)$:

P	Q	R	$P \vee Q$	$P \vee R$	$(P \vee Q) \wedge (P \vee R)$	$P \vee (Q \wedge R)$
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	T	T	T	T
T	F	F	T	T	T	T
F	T	T	T	T	T	T
F	T	F	T	F	F	F
F	F	T	F	T	F	F
F	F	F	F	F	F	F

Since the columns corresponding to $P \vee (Q \wedge R)$ and $(P \vee Q) \wedge (P \vee R)$ are identical, the two statement forms are equivalent. □

Problem 1.8. *Show that $P \wedge (Q \vee R)$ is logically equivalent to $(P \wedge Q) \vee (P \wedge R)$.*

Solution of Problem 1.8. We write a truth table for $P \wedge (Q \vee R)$ and $(P \wedge Q) \vee (P \wedge R)$:

P	Q	R	$P \wedge Q$	$P \wedge R$	$(P \wedge Q) \vee (P \wedge R)$	$P \wedge (Q \vee R)$
T	T	T	T	T	T	T
T	T	F	T	F	T	T
T	F	T	F	T	T	T
T	F	F	F	F	F	F
F	T	T	F	F	F	F
F	T	F	F	F	F	F
F	F	T	F	F	F	F
F	F	F	F	F	F	F

Since the columns corresponding to $P \wedge (Q \vee R)$ and $(P \wedge Q) \vee (P \wedge R)$ are identical, the two statement forms are equivalent. □

Problem 1.9. *Are the statement forms $P \implies (Q \vee R)$ and $(P \implies Q) \vee (P \implies R)$ logically equivalent?*

Solution of Problem 1.9. We could write a truth table for $P \implies (Q \vee R)$ and $(P \implies Q) \vee (P \implies R)$:

P	Q	R	$Q \vee R$	$P \implies Q$	$P \implies R$	$P \implies (Q \vee R)$	$(P \implies Q) \vee (P \implies R)$
T	T	T	T	T	T	T	T
T	T	F	T	T	F	T	T
T	F	T	T	F	T	T	T
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T
F	F	T	T	T	T	T	T
F	F	F	F	T	T	T	T

Since the columns corresponding to $P \implies (Q \vee R)$ and $(P \implies Q) \vee (P \implies R)$ are identical, the two statements are logically equivalent.

An alternate proof would go as follows:

$$P \implies (Q \vee R) \equiv (\neg P) \vee (Q \vee R) \equiv (\neg P) \vee Q \vee R$$

while

$$(P \implies Q) \vee (P \implies R) \equiv ((\neg P) \vee Q) \vee ((\neg P) \vee R)$$
$$\equiv (\neg P) \vee (\neg P) \vee Q \vee R$$
$$\equiv (\neg P) \vee Q \vee R$$

and thus $P \implies (Q \vee R) \equiv (P \implies Q) \vee (P \implies R)$. □

Problem 1.10. *Are the statement forms $P \implies (Q \wedge R)$ and $(P \implies Q) \wedge (P \implies R)$ logically equivalent?*

Hint. *Try to use Problem 1.7.*

Solution of Problem 1.10. We could write a truth table for $P \implies (Q \wedge R)$ and $(P \implies Q) \wedge (P \implies R)$:

P	Q	R	$Q \wedge R$	$P \implies Q$	$P \implies R$	$P \implies (Q \wedge R)$	$(P \implies Q) \wedge (P \implies R)$
T	T	T	T	T	T	T	T
T	T	F	F	T	F	F	F
T	F	T	F	F	T	F	F
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	F	T	T	T	T
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

Since the columns corresponding to $P \implies (Q \wedge R)$ and $(P \implies Q) \wedge (P \implies R)$ are identical, the two statements are logically equivalent.

An alternate proof (assuming we know the result from Problem 1.7) would go as follows:

$$P \implies (Q \wedge R) \equiv (\neg P) \vee (Q \wedge R) \equiv ((\neg P) \vee Q) \wedge ((\neg P) \vee R)$$

while

$$(P \implies Q) \wedge (P \implies R) \equiv ((\neg P) \vee Q) \wedge ((\neg P) \vee R)$$

and thus $P \implies (Q \wedge R) \equiv (P \implies Q) \wedge (P \implies R)$. □

Problem 1.11. *Show that the statement forms $(P \vee Q) \implies R$ and $(P \implies R) \wedge (Q \implies R)$ are logically equivalent.*

Hint. *Try to use DeMorgan's Laws and Problem 1.7.*

Solution of Problem 1.11. We could write a truth table for $(P \vee Q) \implies R$ and $(P \implies R) \wedge (Q \implies R)$:

P	Q	R	$P \vee Q$	$Q \implies R$	$P \implies R$	$(P \vee Q) \implies R$	$(P \implies R) \wedge (Q \implies R)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	T	T	T	T	T
T	F	F	T	T	F	F	F
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	F
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

Since the columns corresponding to $(P \vee Q) \implies R$ and $(P \implies R) \wedge (Q \implies R)$ are identical, the two statements are logically equivalent.

An alternate proof (assuming we know DeMorgan's laws and the result from Problem 1.7) would go as follows:

$$
\begin{aligned}
(P \vee Q) \implies R &\equiv (\neg(P \vee Q) \vee R) \equiv ((\neg P) \wedge \neg Q) \vee R \\
&\equiv ((\neg P) \vee R) \wedge ((\neg Q) \vee R)
\end{aligned}
$$

while

$$(P \implies R) \wedge (Q \implies R) \equiv ((\neg P) \vee R) \wedge ((\neg Q) \vee R)$$

and thus $(P \vee Q) \implies R \equiv (P \implies Q) \wedge (P \implies R)$. □

Problem 1.12. *For all the statement forms below write a logically equivalent statement form that involves only the logical connective $\neg$ and $\vee$.*

1. $P \vee (Q \wedge R)$

2. $(P \vee Q) \wedge (P \vee R)$

3. $P \iff Q$

Hint. *Try to use DeMorgan's laws.*

Solution of Problem 1.12. 1. $P \vee (Q \wedge R) \equiv P \vee \neg((\neg Q) \vee (\neg R))$

2. $(P \vee Q) \wedge (P \vee R) \equiv \neg((\neg(P \vee Q)) \vee \neg(P \vee R))$

3. $P \iff Q \equiv (P \implies Q) \wedge (Q \implies P) \equiv ((\neg P) \vee Q) \wedge ((\neg Q) \vee P) \equiv \neg((\neg((\neg P) \vee Q)) \vee \neg((\neg Q) \vee P))$

□

Problem 1.13. *For all the statement forms below write a logically equivalent statement form that involves only the logical connective* $\neg$ *and* $\wedge$.

1. $P \wedge (Q \vee R)$

2. $(P \wedge Q) \vee (P \wedge R)$

3. $P \iff Q$

Hint. *Try to use DeMorgan's laws.*

Solution of Problem 1.13. 1. $P \wedge (Q \vee R) \equiv P \wedge \neg((\neg Q) \wedge (\neg R))$

2. $(P \wedge Q) \vee (P \wedge R) \equiv \neg((\neg(P \wedge Q)) \wedge \neg(P \wedge R))$

3. $P \iff Q \equiv (P \implies Q) \wedge (Q \implies P) \equiv ((\neg P) \vee Q) \wedge ((\neg Q) \vee P) \equiv (\neg((\neg\neg P) \wedge \neg Q)) \wedge \neg((\neg\neg Q) \wedge \neg P) \equiv (\neg(P \wedge \neg Q)) \wedge \neg(Q \wedge \neg P)$

□

Problem 1.14. *Are the statement forms* $[(\neg P) \implies [Q \wedge \neg Q]]$ *and* P *logically equivalent?*

2 Quantifiers

Problem 2.1. *Let* $x_0 \in (a,b)$, $\ell \in \mathbb{R}$ *and* $f\colon (a,x_0) \cup (x_0,b) \to \mathbb{R}$. *We say that* ℓ *is the limit of* f *at* x_0, *and we write* $\lim_{x \to x_0} f(x) = \ell$, *if for all* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that if* x *satisfies* $0 < |x - x_0| < \delta$ *then* $|f(x) - \ell| < \varepsilon$. *Formally,*

$$\lim_{x \to x_0} f(x) = \ell \iff (\forall \varepsilon > 0)(\exists \delta > 0)(\forall x)[0 < |x - x_0| < \delta \implies |f(x) - \ell| < \varepsilon].$$

Negate the statement $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x)[0 < |x - x_0| < \delta \implies |f(x) - \ell| < \varepsilon]$.

Hint. Remember that $\neg(P \implies Q) \equiv P \wedge \neg Q$. □

Solution of Problem 2.1.

$$(\exists \varepsilon > 0)(\forall \delta > 0)(\exists x)[(0 < |x - x_0| < \delta) \wedge (|f(x) - \ell| \geqslant \varepsilon)].$$

□

Problem 2.2.

1. *Give a possible definition of even numbers using logical symbols, quantifiers, and only the multiplication operation.*
2. *Negate the definition you gave above.*

Solution of Problem 2.2.

1.
$$x \text{ is even } \iff (x \in \mathbb{Z}) \wedge (\exists k \in \mathbb{Z})(x = 2k)$$

2.
$$x \text{ is not even } \iff (x \notin \mathbb{Z}) \vee (\forall k \in \mathbb{Z})(x \neq 2k)$$

□

Problem 2.3.

1. *Give a possible definition of a prime number using logical symbols, quantifiers, and only the multiplication operation.*
2. *Negate the definition you gave above.*

Solution of Problem 2.3.

1.
$$x \text{ is a prime number } \iff [(x \in \mathbb{N}) \wedge (x \neq 1) \wedge [(\forall m \in \mathbb{N})(\forall n \in \mathbb{N})[x = mn \implies ((m = 1) \vee (n = 1))]].$$

2.
$$x \text{ is not a prime number } \iff [(x \notin \mathbb{N}) \vee (x = 1) \vee [(\exists m \in \mathbb{N})(\exists n \in \mathbb{N})(x = mn) \wedge (m > 1) \wedge (n > 1)].$$

□

Problem 2.4. *Write a formal mathematical expression that expresses the fact that a given sequence $(x_n)_{n\in\mathbb{N}}$ does not have a real limit.*

Solution of Problem 2.4.

$$(\forall \ell \in \mathbb{R})(\exists \varepsilon > 0)(\forall N \in \mathbb{N})(\exists n \geqslant N)(|x_n - \ell| \geqslant \varepsilon)$$

□

Problem 2.5. *Negate the statement $P : (\forall n \in \mathbb{Z})(\exists k \in \mathbb{Z})(n^2 + n + 1 = 2k)$. Try to explain what P and $\neg P$ mean.*

Solution of Problem 2.5. $\neg P : (\exists n \in \mathbb{Z})(\forall k \in \mathbb{Z})(n^2 + n + 1 \neq 2k)$. P means that for every integer n the integer $n^2 + n + 1$ is even. $\neg P$ means that there exists an integer n such that the integer $n^2 + n + 1$ is not even. □

Problem 2.6. *Let f be a function from $\mathbb{R}$ to $\mathbb{R}$. We say that f is strictly increasing if*

$$(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})[(x < y) \implies (f(x) < f(y))].$$

Negate the statement above.

Solution of Problem 2.6.

$$(\exists x \in \mathbb{R})(\exists y \in \mathbb{R})[(x < y) \wedge (f(x) \geqslant f(y))].$$

□

Problem 2.7. *Let f be a function from $\mathbb{R}$ to $\mathbb{R}$. Define what it means for f to be strictly decreasing*

Solution of Problem 2.7.

$$f \text{ is strictly decreasing } \iff (\forall x \in \mathbb{R})(\forall y \in \mathbb{R})[(x < y) \implies (f(x) > f(y))].$$

□

Problem 2.8. *Let f be a function from $\mathbb{R}$ to $\mathbb{R}$. Write a formal mathematical expression which expresses the fact that it is not true that f is strictly decreasing or strictly increasing.*

Solution of Problem 2.8. If it is not true that f is strictly decreasing or strictly increasing then f is not strictly decreasing and not strictly decreasing. Formally, it can be expressed with the following statement:

$$[(\exists x \in \mathbb{R})(\exists y \in \mathbb{R})[(x < y) \wedge (f(x) \leqslant f(y))]] \wedge [(\exists w \in \mathbb{R})(\exists z \in \mathbb{R})[(w < z) \wedge (f(w) \geqslant f(z))]].$$

□

Problem 2.9. *Define formally what it means that an integer k divides an integer n.*

Solution of Problem 2.9.

$$k \text{ divides } n \iff (\exists m \in \mathbb{Z})(n = km)$$

□

Problem 2.10. *Give a formal definition of what it means for a number x to be a rational number.*

Solution of Problem 2.11.

$$x \text{ is rational } \iff (x \in \mathbb{R}) \wedge [(\exists p \in \mathbb{Z})(\exists q \in \mathbb{N})\, x = \frac{p}{q}]$$

□

Problem 2.11. *Give a formal definition of what it means for a number x to be a irrational number.*

Solution of Problem 2.11. Since x is irrational if and only if it is not rational,

$$x \text{ is irrational } \iff (x \notin \mathbb{R}) \vee [(\forall p \in \mathbb{Z})(\forall q \in \mathbb{N})(x \neq \frac{p}{q})].$$

□

Problem 2.12. *What is the truth value of the statement $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(\forall z \in \mathbb{R})[xy = xz]$?*

Solution of Problem 2.12. The statement is false, i.e., $(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})(\exists z \in \mathbb{R})[xy \neq xz]$ is true. To see this, let $x = 1$, and let $y \in \mathbb{R}$ be given. If $y \neq 0$, put $z = 2y$. Then $xy = y \neq 2y = xz$. If now $y = 0$, put $z = 1$. Then $xy = 0 \neq 1 = xz$. □

Problem 2.13. *What is the truth value of the statement $(\exists y \in \mathbb{R})(\forall x \in \mathbb{R})(\exists z \in \mathbb{R})[xy = xz]$?*

Solution of Problem 2.13. The statement is true. To see this, put $y = 1$, and let $x \in \mathbb{R}$ be given. Put $z = 1$. Then we have $xy = x = xz$. □

3 Proofs

Problem 3.1. *Prove that the equation* $(E)\colon\ 7x-2=0$ *has a unique solution in* $\mathbb{R}$.

Solution of Problem 3.1. Let $x_0=\frac{2}{7}$. Then $x_0\in\mathbb{R}$ and $7x_0-2=7\frac{2}{7}-2=2-2=0$, so the equation (E) has a solution. Assume now that y is a real solution to the equation (E), then $7y-2=0$ and thus $y=\frac{2}{7}=x_0$. Therefore, the equation (E) has a unique solution. □

Problem 3.2. *Prove that the equation* $(E)\colon\ -3x+8=0$ *has a unique solution in* $\mathbb{R}$.

Solution of Problem 3.2. Assume that y and z are real solutions to the equation (E), then $-3y+8=-3z+8$ and thus $-3y=-3z$. Since $-3\neq 0$ it follows that $y=z$. Therefore, the equation (E) has at most one solution. Now, let $x=\frac{8}{3}$. Then $x\in\mathbb{R}$ and $-3x+8=-3\frac{8}{3}+8=-8+8=0$, so the equation (E) has at least solution. We can thus conclude that the equation (E) at a unique solution. □

Problem 3.3. *Let* $a,b,c\in\mathbb{R}$ *with* $a\neq 0$*. Prove that the equation* $(E)\colon\ ax+b=c$ *has a unique solution in* $\mathbb{R}$.

Solution of Problem 3.3. Let $x_0=\frac{c-b}{a}$. Then $x_0\in\mathbb{R}$ and $ax_0+b=a\frac{c-b}{a}+b=c-b+b=c$, and hence the equation (E) has a solution. Assume now that y and z are real solutions to the equation (E), then $ay+b=c$ and $az+b=c$. Therefore,

$$0=c-c=(ay+b)-(az+b)=a(y-z).$$

Since $a\neq 0$, one has $y-z=0$, and thus $y=z$. The equation (E) has a unique solution. □

Problem 3.4. *Let a, b, and c be integers. Prove that for all integers m and n, if a divides b and a divides c, then a divides* $(bm+cn)$.

Solution of Problem 3.4. Let m and n be fixed integers. Assume that a divides b and that a divides c. Then there exist integers j and k such that $b=aj$ and $c=ak$. (We must show that there exists an integer l such that $bm+cn=al$.) Observe that

$$bm+cn=ajm+akn=a(jm+kn).$$

Put $l=jm+kn$. Then l is an integer such that $bm+cn=al$, and therefore $a\mid(bm+cn)$. Since m and n were fixed but arbitrary, we proved that for all integers m and n, if a divides b and a divides c, then a divides $(bm+cn)$. □

Problem 3.5. *Prove that if* m *and* n *are even, then* $m+n$ *is even.*

Solution of Problem 3.5. Let m and n be fixed integer and assume that m and n are fixed even numbers. Then there exist integers j and k such that $m = 2j$ and $n = 2k$. Let $l = j + k$, then l is an integer, and we have

$$m + n = 2j + 2k = 2(j + k) = 2l.$$

Therefore, $m + n$ is even. Since m and n were fixed but arbitrary even numbers then the proof is complete. □

Problem 3.6. *Prove that if m is even and n is odd, then $m + n$ is odd.*

Solution of Problem 3.6. Assume that m is a fixed even number and n is a fixed odd number. Then there exist integers j and k such that $m = 2j + 1$ and $n = 2k$. Let $l = j + k$, then l is an integer, and we have

$$m + n = 2j + 1 + 2k = 2(j + k) + 1 = 2l + 1.$$

Therefore, $m + n$ is odd. The proof is complete since m and n were fixed but arbitrary. □

Problem 3.7. *Prove that for all $m, n \in \mathbb{Z}$, if m is even, then mn is even.*

Solution of Problem 3.7. Let m and n be integers and assume that m is even. Then there exists and integer k such that $m = 2k$. Put $l = kn$. Then l is an integer, and we have $mn = 2kn = 2l$. Therefore, mn is even. □

Problem 3.8. *Show that for all $n \in \mathbb{Z}$, $4n + 7$ is odd.*

Solution of Problem 3.8. Fix $n \in \mathbb{Z}$. Let $k = 2n + 3$, then k is an integer, and we have

$$4n + 7 = 2(2n) + 2(3) + 1 = 2(2n + 3) + 1 = 2k + 1.$$

Therefore, $4n + 7$ is odd. Since n was fixed but arbitrary the conclusion follows. □

Problem 3.9. *Let n be an integer. Prove that if n^2 is even, then n is even.*

Solution of Problem 3.9. Let $n \in \mathbb{Z}$ and assume n is not even. Then n is odd, hence there is some $k \in \mathbb{Z}$ such that $n = 2k + 1$. Thus $n^2 = (2k + 1)^2 = 2(2k^2 + 2k) + 1 = 2r + 1$, with $r = 2k^2 + 2k \in \mathbb{Z}$, and thus n^2 is odd. Therefore, by contraposition, if n is even, then n^2 is even. □

Problem 3.10. *Let n be an integer. Prove that if n^3 is even, then n is even.*

Solution of Problem 3.10. Let us prove the contrapositive "if n is not even then n^3 is not even", or equivalently "if n is odd then n^3 is odd". Assume that n is odd, then there exists $k \in \mathbb{Z}$ such that $n = 2k + 1$, and hence $n^3 = (2k + 1)^3 = (2k)^3 + 3 \cdot (2k)^2 + 3 \cdot 2k + 1 = 2(4k^3 + 6 \cdot k^2 + 3k) + 1 = 2r + 1$, where $r = 4k^3 + 6 \cdot k^2 + 3k$ is an integer. Therefore n^3 is odd. □

Problem 3.11. *For this problem you can use the following fact that will be proven later: 3 does not divides n if and only if there exists an integer k and an integer $i \in \{1,2\}$ such that $n = 3k+i$.*

Prove that for every integer n, if 3 divides n^2 then 3 divides n.

Solution of Problem 3.12. Let n be an integer and assume that 3 does not divide n. Then there exists an integer k and an integer $i \in \{1,2\}$ such that $n = 3k+i$. Therefore,

$$n^2 = (3k+i)^2 = 9k^2 + 6ki + i^2 = 3(3k^2 + 2ki) + i^2.$$

If $i = 1$ then $i^2 = 1^2 = 1$, and $n^2 = 3r+1$ with $r = 3k^2 + 2ki \in \mathbb{Z}$. Otherwise, if $i = 2$ then $n^2 = 3(3k^2 + 2ki) + 2^2 = 3(3k^2+2ki)+3+1 = 3(3k^2+2ki+1)+1 = 3s+1$, with $s = 3k^2+2ki+1 \in \mathbb{Z}$. In any case, $n^2 = 3t+1$ for some integer t and thus 3 does not divide n^2. □

Problem 3.12. *Prove that there are no integers m and n such that $8m+26n = 1$.*

Solution of Problem 3.12. Assume, towards a contradiction, that there exist integers m and n such that $8m+26n = 1$. Then $1 = 2(4m+13n) = 2r$, with $r = 4m+13n \in \mathbb{Z}$, and 1 would be even. But 1 is odd, a contradiction. Thus, there are no integers m and n such that $8m+26n = 1$. □

Problem 3.13. *Are there integers m and n such that $m^2 = 4n+3$?*

Solution of Problem 3.13. Assume, towards a contradiction, that there exist integers m and n such that $m^2 = 4n+3$. Then $m^2 = 2(2n+1)+1$, so m^2 is odd, and therefore m is odd. Thus there is some $k \in \mathbb{Z}$ such that $m = 2k+1$, and we have

$$4n+3 = (2k+1)^2 = 4k^2+4k+1,$$

so $4n+2 = 4k^2+4k$, and thus $2n+1 = 2k^2+2k = 2(k^2+k)$. Hence $2n+1$ is both even and odd, a contradiction. Therefore, there do not exist integers m and n such that $m^2 = 4n+3$. □

Problem 3.14. *Let $x \in \mathbb{R}$. Show that if for all $\varepsilon > 0$, $|x| < 2\varepsilon$, then $x = 0$.*

Solution of Problem 3.14. Let $x \in \mathbb{R}$. Assume that for all $\varepsilon > 0$, $|x| < 2\varepsilon$ and for the sake of a contradiction assume that $x \neq 0$. If we put $\varepsilon_0 = \frac{|x|}{4}$ then $\varepsilon_0 > 0$. Therefore, $|x| < 2\varepsilon_0 = 2\frac{|x|}{4}$. Since $|x| \neq 0$, it follows that $|x| < \frac{|x|}{2}$ and thus $1 < \frac{1}{2}$, a contradiction. □

Problem 3.15. *Prove that $\sqrt[3]{2}$ is irrational.*

Hint. Use Problem 3.10. □

Solution of Problem 3.15. Assume by contradiction that $\sqrt[3]{2}$ is rational and write, as we may, $\sqrt[3]{2} = \frac{p}{q}$ with $p \in \mathbb{Z}$, $q \in \mathbb{Z}$, $q > 0$, where p and q have no common factors. Thus, $2 = (\frac{p}{q})^3$ and $2q^3 = p^3$, which means that p^3 is even. By Problem 3.10 above p is even, and there exists $k \in \mathbb{Z}$ such that $p = 2k$. It follows that $q^3 = \frac{p^3}{2} = \frac{8k^3}{2} = 2 \cdot 2k^3$ and hence q^3 is even. By Problem 3.10, q is even, a contradiction. Indeed, p and q being even they have 2 as a common factor which contradicts our assumption. □

Problem 3.16. *Show that* $\sqrt{3}$ *is irrational.*

Hint. Use Problem 3.11. □

Solution of Problem 3.16. Assume, towards a contradiction, that $\sqrt{3}$ is rational. Then there exist nonnegative integers m and n, with $n \neq 0$, such that $\sqrt{3} = \frac{m}{n}$. Without loss of generality, we may assume that m and n have no common factors. Then $m^2 = 3n^2$, so 3 divides m^2, and by the result of Problem 3.11 it follows that 3 divides m. Thus there is some $k \in \mathbb{Z}$ such that $m = 3k$. Hence we have

$$9k^2 = m^2 = 3n^2,$$

so $n^2 = 3k^2$. Thus 3 divides n^2, and applying the results of Problem 3.11 again, it follows that 3 divides n. But then both m and n are divisible by 3, so they share a common factor, contradicting our assumption. Therefore, $\sqrt{3}$ is irrational. □

Problem 3.17. *Show that* $\log(3)$ *is irrational.*

Hint. You can use the the following property of the log function: $x = \log(3) \iff 2^x = 3$ (no proof needed) You can also use the binomial formula (no proof needed). Everything else that you might need needs to be proven. □

Problem 3.18. *Prove that for all real numbers x and y with* $y \geqslant 0$*, if* $x^2 \geqslant 4y$*, then* $x \geqslant 2\sqrt{y}$ *or* $x \leqslant -2\sqrt{y}$*.*

Solution of Problem 3.18. Let $x \in \mathbb{R}$ and $y > 0$ and assume that $x^2 \geqslant 4y$. Either $x \geqslant 2\sqrt{y}$ and the conclusion follows, or $x < 2\sqrt{y}$ but then $0 \leqslant x^2 - 4y = (x - 2\sqrt{y})(x + 2\sqrt{y})$ and hence $x + 2\sqrt{y} \leqslant 0$, i.e. $x \leqslant -2\sqrt{y}$, and the conclusion follows. □

Problem 3.19. *Prove that for all integers k,* $k(k+3)$ *is even.*

Solution of Problem 3.19. Let $k \in \mathbb{Z}$.

Case 1 k is even and there exists $n \in \mathbb{Z}$ such that $k = 2n$. So, $k(k+3) = 2n(2n+3)$ which is even.

Case 2 k is odd and there exists $n \in \mathbb{Z}$ such that $k = 2n+1$. So, $k(k+3) = (2n+1)(2n+4) = 2(2n+1)(n+2)$ which is even.

□

Problem 3.20. *Prove that for all integers k,* $(k+1)(k+6)$ *is even.*

Solution of Problem 3.20. Let $k \in \mathbb{Z}$. Either k is even and there exists $n \in \mathbb{Z}$ such that $k = 2n$, and $(k+1)(k+6) = (2n+1)(2n+6) = 2(2n+1)(n+3)$ which is even, or k is odd and there exists $n \in \mathbb{Z}$ such that $k = 2n+1$, and thus $(k+1)(k+6) = (2n+2)(2n+7) = 2(n+1)(2n+7)$ is even. □

For the following problems we recall the definition of the absolute value function

$$|x| := \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$$

Problem 3.21. *Show that for all* $x \in \mathbb{R}$, $|x| \geqslant 0$ *with* $|x| = 0$ *if and only if* $x = 0$.

Proof of Problem 3.21. We prove the first part of the statement. Let $x \in \mathbb{R}$. Then, either $x \geqslant 0$ or $x < 0$. If $x \geqslant 0$ then by definition $|x| = x \geqslant 0$. Otherwise, if $x < 0$ then by definition $|x| = -x > 0$. For the equivalence in the second part, if $x = 0$ then by definition $|x| = 0$. If $|x| = 0$ then by definition $|x| = x$ and thus $x = 0$. □

Problem 3.22. *Prove that for all real numbers x and y,* $|x-y| = |y-x|$.

Proof of Problem 3.22. Let $x, y \in \mathbb{R}$. In the case $x - y \geqslant 0$ then $y - x \leqslant 0$ and $|x-y| = x-y$, but $|y-x| = -(y-x) = x-y$ and thus $|x-y| = |y-x|$. In the case $x-y < 0$ then $y-x > 0$ and $|x-y| = -(x-y) = y-x$, but $|y-x| = y-x$ and thus $|x-y| = |y-x|$. Therefore in all cases $|x-y| = |y-x|$. □

Problem 3.23. *Prove that for all real numbers x and y,* $|xy| = |x||y|$.

Problem 3.24. *Let* $x \in \mathbb{R}$ *and* $M \geqslant 0$. *Show that* $|x| \leqslant M \iff -M \leqslant x \leqslant M$.

Proof of Problem 3.24. Let $x \in \mathbb{R}$ and $M \geqslant 0$.

Proof of $\implies$: Assume that $|x| \leqslant M$, then if $x \geqslant 0$, then $x = |x|$ and $-M \leqslant 0 \leqslant x = |x| \leqslant M$. Otherwise, if $x < 0$ then $|x| = -x$ and $-M \leqslant 0 < -x = |x| < M$.

Proof of $\impliedby$: Assume that $-M \leqslant x \leqslant M$. In the case $x \geqslant 0$ then $|x| = x$, but $x \leqslant M$ and it follows that $|x| \leqslant M$. In the case $x < 0$ then $|x| = -x$, but since $-M \leqslant x$ then $-x \leqslant M$ and hence $|x| \leqslant M$

□

Problem 3.25. *Prove that for all real numbers x and y,* $|x+y| \leqslant |x| + |y|$.

Hint. You could use Problem 3.24. ☐

Proof of Problem 3.25. Let $x, y \in \mathbb{R}$. Since $-|x| \leqslant x \leqslant |x|$ and $-|y| \leqslant y \leqslant |y|$ then by adding up theses two inequalities $-(|y|+|x|) \leqslant x+y \leqslant |x|+|y|$ and by Problem 3.24 $|x+y| \leqslant |x|+|y|$. ☐

Problem 3.26. *Prove that for all $x, y, z \in \mathbb{R}$, $|x-y| \leqslant |x-z|+|y-z|$.*

Hint. You could use Problem 3.25. ☐

Proof of Problem 3.26. Let $x, y, z \in \mathbb{R}$ and set $a = x - z$ and $b = z - y$. It follows from Problem 3.25 that $|x-y| = |a+b| \leqslant |a|+|b| = |x-z|+|y-z|$.

☐

Problem 3.27. *Prove that for all real numbers x and y, $\big||x|-|y|\big| \leqslant |x-y|$.*

Hint. You could use Problem 3.25. ☐

Proof of Problem 3.27. Let $x, y \in \mathbb{R}$, then by Problem 3.25 $|x| = |x-y+y| \leqslant |x-y|+|y|$, and $|y| = |y-x+x| \leqslant |y-x|+|x|$. Thus $|x|-|y| \leqslant |x-y|$ and $|y|-|x| \leqslant |x-y|$ and the conclusion follows. ☐

Problem 3.28. *Let x, y be real numbers. Show that*

$$\forall \varepsilon > 0,\ x < y+\varepsilon \iff x \leqslant y.$$

Proof of Problem 3.28. Proof of $\Longleftarrow$: Assume that $x \leqslant y$, then if $\varepsilon > 0$ it follows that $x < y+\varepsilon$. Proof of $\Longrightarrow$: Assume that $x < y+\varepsilon$ for all $\varepsilon > 0$. Assume by contradiction that $x > y$ and let $\varepsilon_0 = x-y > 0$. By our assumption, $x < y+\varepsilon_0 = y+(x-y) = x$; a contradiction. ☐

Problem 3.29. *Let x, y be real numbers. Show that $x > y-\varepsilon$ for all $\varepsilon > 0$ if and only if $x \geqslant y$.*

Proof of Problem 3.29. Assume that $x < y+\varepsilon$ for all $\varepsilon > 0$. Assume by contradiction that $x > y$ and let $\varepsilon_0 = x-y > 0$. By our assumption, $x < y+\varepsilon_0 = y+(x-y) = x$; a contradiction. For the other direction, if $x \geqslant y$ and $\varepsilon > 0$ then $x > y-\varepsilon$. ☐

Problem 3.30. *Prove that for all real numbers x and y, if $x < y$, then $x < \frac{x+y}{2} < y$.*

Solution of Problem 3.30. Let x and y be real numbers such that $x < y$. Then $2x = x+x < x+y$, and thus $x < \frac{x+y}{2}$. Similarly, $x+y < y+y = 2y$, and thus $\frac{x+y}{2} < y$. Combining our results, we obtain

$$x < \frac{x+y}{2} < y$$

as desired. ☐

Problem 3.31. *Prove that for all positive real numbers x, the sum of x and its reciprocal is greater than* 2*.*

Solution of Problem 3.31. Let x be a real number such that $x > 0$. Observe that $0 \leqslant (x-1)^2 = x^2 - 2x + 1$, so $2x \leqslant x^2 + 1$ and thus since $x > 0$ one has

$$2 = \frac{1}{x} \cdot 2x \leqslant \frac{1}{x} \cdot (x^2 + 1) = x + \frac{1}{x}$$

as desired. □

Problem 3.32. *1. Prove that for all $x, y \in \mathbb{R}^+$, $\sqrt{xy} \leqslant \frac{x+y}{2}$.*

2. Show that that for all $x, y \in \mathbb{R}^+$, $\sqrt{xy} = \frac{x+y}{2}$ if and only if $x = y$

Solution of Problem 3.32.

1. Let $x, y \in \mathbb{R}^+$ be given. Observe that $0 \leqslant (x-y)^2 = x^2 - 2xy + y^2$, and so $4xy \leqslant x^2 + 2xy + y^2 = (x+y)^2$. Thus $2\sqrt{xy} \leqslant x + y$, and therefore $\sqrt{xy} \leqslant \frac{x+y}{2}$.

2. Assuming $x = y$, we obtain
$$\sqrt{xy} = \sqrt{x^2} = x = \frac{2x}{2} = \frac{x+y}{2}.$$
Conversely, assume $\sqrt{xy} = \frac{x+y}{2}$. Then $4xy = (x+y)^2 = x^2 + 2xy + y^2$, and rearranging we obtain $0 = x^2 - 2xy + y^2 = (x-y)^2$, so $0 = x - y$ and therefore $x = y$.

□

4 Applications of the Principle of Mathematical Induction

Problem 4.1. *Prove that for all integers $n \geqslant 1$,*

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Solution of Problem 4.1. First observe that for $n = 1$ we have

$$\frac{n(n+1)(2n+1)}{6} = \frac{1 \cdot 2 \cdot 3}{6} = 1,$$

but $\sum_{k=1}^{1} k^2 = 1^2 = 1$ and thus the equality holds if $n = 1$. Now let $n \geqslant 1$ and assume that assume $\sum_{k=1}^{n} k^2 =$

$\frac{n(n+1)(2n+1)}{6}$. Then we have

$$\begin{aligned}\sum_{k=1}^{n+1} k^2 &= (n+1)^2 + \sum_{k=1}^{n} k^2 \\ &= (n+1)^2 + \frac{n(n+1)(2n+1)}{6} \\ &= \frac{6(n+1)(n+1)}{6} + \frac{n(n+1)(2n+1)}{6} \\ &= \frac{(n+1)[6(n+1)+n(2n+1)]}{6} \\ &= \frac{(n+1)(2n^2+7n+6)}{6} \\ &= \frac{(n+1)(n+2)(2n+3)}{6} \\ &= \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6},\end{aligned}$$

so the equality holds for $n+1$. Therefore, it follows by the Principle of Mathematical Induction that for all integers $n \geqslant 1$, $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$. □

Problem 4.2. *Prove that for all integers $n \geqslant 0$,*

$$\sum_{k=0}^{n} 2^k = 2^{n+1} - 1.$$

Solution of Problem 4.2. First observe that for $n = 0$, we have $2^0 = 1 = 2^{0+1} - 1$, so the equality holds in this case. Let $n \geqslant 0$ and assume that the equality holds for n, i.e., $\sum_{k=0}^{n} 2^k = 2^{n+1} - 1$. Then we have

$$\sum_{k=0}^{n+1} 2^k = 2^{n+1} + \sum_{k=0}^{n} 2^k = 2^{n+1} + 2^{n+1} - 1 = 2 \cdot 2^{n+1} - 1 = 2^{(n+1)+1} - 1$$

so the equality holds for $n+1$. Therefore, it follows by the Principle of Mathematical Induction that for all integers $n \geqslant 0$, $\sum_{k=0}^{n} 2^k = 2^{n+1} - 1$. □

Problem 4.3. *Prove that for all integers $n \geqslant 1$,*

$$\sum_{k=1}^{n} (2k-1) = n^2.$$

Solution of Problem 4.3. First observe that for $n = 1$, we have $1^2 = 1 = 2 \cdot 1 - 1$, so the result is true in this case. Now assume that the result holds for some positive integer m. Then we have

$$\sum_{k=1}^{m+1} (2k-1) = 2(m+1) - 1 + \sum_{k=1}^{m} (2k-1) = m^2 + 2m + 1 = (m+1)^2,$$

so the result is true for $m+1$. Therefore, it follows by induction that the result holds for all positive integers.

□

Problem 4.4. *Prove that for all integers $n \geqslant 1$,*

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}.$$

Solution of Problem 4.4. For $n = 1$, $\sum_{k=1}^{1} \frac{1}{k(k+1)} = \frac{1}{1(1+1)} = \frac{1}{2} = \frac{1}{1+1}$, and the equality holds. Assume that the result holds for some positive integer m. Then we have

$$\begin{aligned}
\sum_{k=1}^{m+1} \frac{1}{k(k+1)} &= \frac{1}{(m+1)(m+2)} + \sum_{k=1}^{m} \frac{1}{k(k+1)} \\
&= \frac{1}{(m+1)(m+2)} + \frac{m}{m+1} \\
&= \frac{1}{(m+1)(m+2)} + \frac{m^2+2m}{(m+1)(m+2)} \\
&= \frac{(m+1)^2}{(m+1)(m+2)} \\
&= \frac{(m+1)}{(m+1)+1}
\end{aligned}$$

so the result is true for $m+1$. Therefore, it follows by induction that the result holds for all positive integers.

□

Problem 4.5. *Prove that for all integers $n \geqslant 1$,*

$$\sum_{k+1}^{n} (2k-1)^2 = \frac{4n^3-n}{3}.$$

Solution of Problem 4.5. First observe that

$$\frac{4 \cdot 1^3 - 1}{3} = 1 = (2 \cdot 1 - 1)^2,$$

so the result is true for $n = 1$. Now assume that the result holds for some positive integer m. Then we have

$$\begin{aligned}
\sum_{k=1}^{m+1} (2k-1)^2 &= (2m+1)^2 + \frac{4m^3-m}{3} \\
&= \frac{12m^2+12m+3}{3} + \frac{4m^3-m}{3} \\
&= \frac{4m^3+12m^2+12m+3-m}{3} \\
&= \frac{4(m^3+3m^2+3m+1)-(m+1)}{3} \\
&= \frac{4(m+1)^3-(m+1)}{3},
\end{aligned}$$

so the result is true for $m+1$. Therefore, it follows by induction that the result holds for all positive integers.

□

Problem 4.6. *Conjecture a formula for* $\sum_{k=1}^{n}(-1)^k k^2$*, for all* $n \geqslant 1$ *and then prove the formula is correct using induction.*

Solution of Problem 4.6. After calculating a few iterates, you should see that for all $n \geqslant 1$ the formula you are looking for is

$$\sum_{k=1}^{n}(-1)^k k^2 = (-1)^n\left(\frac{n(n+1)}{2}\right).$$

Indeed, this holds for $n = 1$. Now assume the formula holds for some positive integer m. Then we have

$$\begin{aligned}\sum_{k=1}^{m+1}(-1)^k k^2 &= (-1)^{m+1}(m+1)^2 + (-1)^m\left(\frac{m(m+1)}{2}\right)\\ &= \frac{(-1)^{m+1}}{2}\left(2m^2+4m+2-m^2-m\right)\\ &= \frac{(-1)^{m+1}}{2}\left(m^2+3m+2\right)\\ &= (-1)^{m+1}\left(\frac{(m+1)((m+1)+1)}{2}\right),\end{aligned}$$

so the formula holds for $m+1$. Therefore, by induction, the result holds for all positive integers. □

Problem 4.7. *Prove that for all integers* $n \geqslant 1$*,* $n < 10^n$*.*

Solution of Problem 4.7. The result is true for $n = 1$. Now assume the result holds for some positive integer m. Then we have

$$m+1 < 10^m + 1 < 9\cdot 10^m + 10^m = 10\cdot 10^m = 10^{m+1},$$

so the result is true for $m+1$. Therefore, by induction it follows that the result is true for all positive integers. □

Problem 4.8. *Prove that for all integers* $n \geqslant 7$*,* $\left(\frac{4}{3}\right)^n > n$*.*

Solution of Problem 4.8. Direct calculation show that $\left(\frac{4}{3}\right)^7 > 7$. Now assume that for some $m \geqslant 7$ we have $\left(\frac{4}{3}\right)^m > m$. Then $3 < 7 \leqslant m < \left(\frac{4}{3}\right)^m$, so $1 < \frac{1}{3}\left(\frac{4}{3}\right)^m$ and thus

$$m+1 < \left(\frac{4}{3}\right)^m + \frac{1}{3}\left(\frac{4}{3}\right)^m = \left(\frac{4}{3}\right)^{m+1},$$

so the result holds for $m+1$. Therefore, by induction it follows that the result holds for all positive integers $n \geqslant 7$. □

Problem 4.9. *Prove that for all integers* $n \geqslant 1$*,* $n^3 + 8n + 9$ *is divisible by* 3.

Solution of Problem 4.9. First observe that $1^3+8\cdot 1+9=18=3\cdot 6$, so the result is true for $n=1$. Now assume that the result holds for some positive integer m. Then there is some integer k such that $m^3+8m+9=3k$, and we have

$$\begin{aligned}(m+1)^3+8(m+1)+9 &= (m^3+3m^2+3m+1)+(8m+8)+9\\ &= (m^3+8m+9)+3m^2+3m+9\\ &= 3(k+m^2+m+3),\end{aligned}$$

so the result is true for $m+1$. Therefore, by induction it follows that the result holds for all positive integers.

□

Problem 4.10. *Prove that for all integers $n \geqslant 1$, $3^{2n}-1$ is divisible by 8.*

Solution of Problem 4.10. First observe that $3^{2\cdot 1}-1=8\cdot 1$, so the result is true for $n=1$. Now assume the result is true for some positive integer m. Then there is some integer k such that $3^{2m}-1=8k$, and we have

$$\begin{aligned}3^{2(m+1)}-1 &= 3^{2m+2}-1\\ &= 9\cdot 3^{2m}-1\\ &= 9\cdot(3^{2m}-1)+8\\ &= 8\cdot(9k+1),\end{aligned}$$

so the result is true for m_1. Therefore, by induction it follows that the result is true for all positive integers.

□

Problem 4.11. *Prove that for all integers $n \geqslant 5$, $n^2 < 2^n$.*

Solution of Problem 4.11. Let $P(n)$ be the statement $n^2 < 2^n$. Observe that $5^2 = 25 < 32 = 2^5$, so $P(5)$ is true. Now assume that $P(m)$ is true for some $m \geqslant 5$. Since $m \geqslant 5$, we have $2m+1 < 3m < m^2$, so that

$$(m+1)^2 = m^2+2m+1 < 2m^2 < 2\cdot 2^m = 2^{m+1}.$$

and thus $P(m+1)$ is true. Therefore, by induction it follows that $P(n)$ is true for all $n \geqslant 5$. □

Problem 4.12. *Prove that for all integers $n \geqslant 4$, $2^n < n!$.*

Solution of Problem 4.12. Clearly $2^4 = 16 < 24 = 4!$. Now assume that $2^m < m!$ for some positive integer $m \geqslant 4$. Then we have

$$2^{m+1} = 2\cdot 2^m < 2m! < (m+1)\cdot m! = (m+1)!$$

Therefore, it follows by induction that $2^n < n!$ for all positive integers n. □

Problem 4.13. *Assuming that* $(1+\frac{1}{n})^n < e$*, for all* $n \geqslant 1$*, prove that for all* $n \geqslant 1$*,* $n! > (\frac{n}{e})^n$*.*

Problem 4.14. *Show that for all* $n \geqslant 12$ *there exist* $x_n \in \mathbb{Z}$ *and* $y_n \in \mathbb{Z}$ *such that* $n = 3x_n + 7y_n$

Problem 4.15. *Prove that for all positive integers n,* $4^n - 1$ *is divisible by* 3*.*

Solution of Problem 4.15. First note that $4^1 - 1 = 3$, and thus $3 \mid 4^1 - 1$. Now assume that $4^m - 1$ is divisible by 3 for some positive integer m. Then $4^m - 1 = 3k$ for some $k \in \mathbb{Z}$, hence we have

$$4^{m+1} - 1 = 4(4^m - 1) + 3 = 3(4k+1).$$

Thus $4^{m+1} - 1$ is divisible by 3, and it follows by induction that $4^n - 1$ is divisible by 3 for all positive integers n. □

Problem 4.16. *Let* $a_1 = 2$*, and let* $a_{n+1} = \frac{1}{2}(a_n + 3)$ *for all* $n \geqslant 1$*.*

(a) Prove that for all positive integers n, $a_n < a_{n+1}$*.*

(b) Prove that for all positive integers n, $a_n < 3$*.*

(c) Prove that for all positive integers n, $a_n = 3 - \frac{1}{2^{n-1}}$*.*

Solution of Problem 4.16. (a) First note that $a_2 = \frac{1}{2}(a_1 + 3) = \frac{1}{2}(2+3) = \frac{5}{2} > a_1$. Now assume that $a_{m+1} > a_m$ for some positive integer m. Then we have

$$a_{m+2} = \frac{1}{2}(a_{m+1} + 3) > \frac{1}{2}(a_m + 3) = a_{m+1}.$$

By induction, it follows that $a_{n+1} > a_n$ for all positive integers n.

(b) Clearly $a_1 = 2 < 3$. Now assume that $a_m < 3$ for some positive integer m. Then we have

$$a_{m+1} = \frac{1}{2}(a_m + 3) < \frac{1}{2}(3+3) = 3.$$

By induction, it follows that $a_n < 3$ for all positive integers n.

(c) Clearly $a_1 = 2 = 3 - \frac{1}{2^{1-1}}$. Now assume that $a_m = 3 - \frac{1}{2^{m-1}}$ for some positive integer m. Then we have

$$a_{m+1} = \frac{1}{2}(a_m + 3) = \frac{1}{2}\left(6 - \frac{1}{2^{m-1}}\right) = 3 - \frac{1}{2^{m+1-1}}.$$

By induction, it follows that $a_n = 3 - \frac{1}{2^{n-1}}$ for all positive integers n.

□

Problem 4.17. *Let $r \in \mathbb{R}$ with $r \neq 1$. Prove that*

$$\sum_{k=0}^{n-1} r^k = \frac{1-r^n}{1-r}.$$

Solution of Problem 4.17. Fix $r \in \mathbb{R}$ with $r \neq 1$. Note that $\sum_{k=0}^{1-1} r^k = 1 = \frac{1-r^1}{1-r}$. Now assume that $\sum_{k=0}^{m-1} r^k = \frac{1-r^m}{1-r}$ for some positive integer n. Then we have

$$\sum_{k=0}^{m} r^k = r^m + \sum_{k=0}^{m-1} r^k = r^m \frac{1-r}{1-r} + \frac{1-r^m}{1-r} = \frac{1-r^{m+1}}{1-r}.$$

Thus, by induction it follows that $\sum_{k=0}^{n-1} r^k = \frac{1-r^n}{1-r}$ is true for all positive integers n. □

Problem 4.18. *Prove Bernoulli's Inequality: Let $x > -1$. Then for all $n \in \mathbb{N}$, $(1+x)^n \geqslant 1+nx$.*

Solution of Problem 4.18. For $n = 1$ equality (hence the inequality) holds. Now suppose that $(1+x)^m \geqslant 1+mx$ for some $m \in \mathbb{N}$. Then we have

$$(1+x)^{m+1} \geqslant (1+x)(1+mx) = 1+mx+x+mx^2 \geqslant 1+(m+1)x.$$

(Note that we used the assumption that $x > 1$ in the first inequality.) Thus by induction, the inequality holds for all $n \in \mathbb{N}$. □

Problem 4.19. *Let $x, y \in \mathbb{R}$. Prove the binomial theorem: for all integers $n \geqslant 1$,*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

Solution of Problem 4.19. First note that

$$(x+y)^1 = x+y = \binom{1}{0} x^1 y^0 + \binom{1}{1} x^0 y^1.$$

Now assume that for some $m \in \mathbb{N}$ we have

$$(x+y)^m = \sum_{k=0}^{m} \binom{m}{k} x^{m-k} y^k.$$

Then we have

$$\begin{aligned}
(x+y)^{m+1} &= (x+y)(x+y)^m \\
&= \sum_{k=0}^{m} \binom{m}{k} x^{m+1-k} y^k + \sum_{k=0}^{m} \binom{m}{k} x^{m-k} y^{k+1} \\
&= \binom{m}{0} x^{m+1} y^0 + \sum_{k=1}^{m} \binom{m}{k} x^{m+1-k} y^k + \sum_{k=0}^{m-1} \binom{m}{k} x^{m-k} y^{k+1} + \binom{m}{m} x^0 y^{m+1} \\
&= \binom{m}{0} x^{m+1} y^0 + \sum_{k=1}^{m} \left(\binom{m}{k} + \binom{m}{k-1} \right) x^{m+1-k} y^k + \binom{m}{m} x^0 y^{m+1} \\
&= \binom{m+1}{0} x^{m+1} y^0 + \sum_{k=1}^{m} \binom{m+1}{k} x^{m+1-k} y^k + \binom{m+1}{m+1} x^0 y^{m+1} \\
&= \sum_{k=0}^{m+1} \binom{m+1}{k} x^{m+1-k} y^k
\end{aligned}$$

as desired. □

Problem 4.20. *Let n be an integer. Show that if n is even then n^k is even for all $k \in \mathbb{N}$.*

5 Applications of the Principle of Strong Mathematical Induction

Problem 5.1. *For $i \in \mathbb{N}$, let p_i denote the ith prime number, so that*

$$p_1 = 2, \qquad p_2 = 3, \qquad p_3 = 5, \ldots.$$

Prove that for all $n \in \mathbb{N}$, $p_n \leqslant 2^{2^{n-1}}$.

Hint. *For the induction step, given $m \in \mathbb{N}$, show that $p_{m+1} \leqslant p_1 p_2 \cdots p_m + 1$.*

Solution of Problem 5.1. First observe that $p_1 = 2 = 2^{2^{1-1}}$. Now fix $m \in \mathbb{N}$, and assume that $p_k \leqslant 2^{2^{k-1}}$ for $1 \leqslant k \leqslant m$. Note that $p_{m+1} \leqslant p_1 p_2 \cdots p_m + 1$, since p_k does not divide $p_1 p_2 \cdots p_m + 1$ for $1 \leqslant k \leqslant m$. Thus, we have

$$p_{m+1} \leqslant p_1 p_2 \cdots p_m + 1 \leqslant 2^{\sum_{k=0}^{m-1} 2^k} + 1 = 2^{2^m - 1} + 1 < 2 \cdot 2^{2^m - 1} = 2^{2^m}.$$

□

Problem 5.2. *Show that the principle of strong mathematical induction implies the principle of mathematical induction.*

Solution of Problem 5.2. Assume the principle of strong mathematical induction, and let $P(n)$ be a statement about the positive integer n. Assume that $P(1)$ is true, and that for all $m \in \mathbb{N}$, if $P(m)$ is true then $P(m+1)$ is true. Let $m \in \mathbb{N}$ be given, and assume that $P(k)$ is true for $1 \leqslant k \leqslant m$. Then $P(m)$ is true, so $P(m+1)$ is true. Thus, by the principle of strong mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$. Therefore, the principle of mathematical induction is true. □

Problem 5.3. *Show that the principle of mathematical induction implies the principle of strong mathematical induction.*

Solution of Problem 5.3. Assume the principle of mathematical induction, and let $P(n)$ be a statement about the positive integer n. Assume that $P(1)$ is true, and that for all $m \in \mathbb{N}$, if $P(k)$ is true for $1 \leqslant k \leqslant m$, then $P(m+1)$ is true. For $n \in \mathbb{N}$, let $Q(n)$ be the statement "$P(k)$ is true for all $k \leqslant n$". Clearly $Q(1)$ is true. If $m \in \mathbb{N}$ and $Q(m)$ is true, then $P(k)$ is true for all $1 \leqslant k \leqslant m$. By assumption $P(m+1)$ is true, and thus $Q(m+1)$ is true. By the principle of mathematical induction, $Q(n)$ is true for all $n \in \mathbb{N}$. But if $Q(n)$ is true, then $P(n)$ is true, and thus $P(n)$ is true for all $n \in \mathbb{N}$. Therefore, the principle of strong mathematical induction is true. □

6 Sequences defined by a recurrence relation

Problem 6.1. *Let $a_1 = 2$, $a_2 = 4$, and $a_{n+1} = 7a_n - 10a_{n-1}$ for all $n \geqslant 2$. Conjecture a closed formula for a_n and prove your result.*

Solution of Problem 6.1. We will show that $a_n = 2^n$ for each $n \in \mathbb{N}$. Indeed, $a_1 = 2 = 2^1$ and $a_2 = 4 = 2^2$. Now assume that for some $m \geqslant 2$ we have $a_k = 2^k$ for $1 \leqslant k \leqslant m$. Then we have

$$a_{m+1} = 7a_m - 10a_{m-1} = 7 \cdot 2^m - 10 \cdot 2^{m-1} = 14 \cdot 2^{m-1} - 10 \cdot 2^{m-1} = 4 \cdot 2^{m-1} = 2^{m+1}.$$

Therefore, by the principle of strong mathematical induction it follows that $a_n = 2^n$ for all $n \in \mathbb{N}$. □

Problem 6.2. *Let $a_1 = 3$, $a_2 = 4$, and $a_{n+1} = \frac{1}{3}(2a_n + a_{n-1})$ for all $n \geqslant 2$. Prove that for all positive integers n, $3 \leqslant a_n \leqslant 4$.*

Solution of Problem 6.2. Clearly $3 \leqslant a_1 \leqslant 4$ and $3 \leqslant a_2 \leqslant 4$. Now fix $m \geqslant 2$, and assume that $3 \leqslant a_k \leqslant 4$ for all $1 \leqslant k \leqslant m$. Then we have

$$a_{m+1} = \frac{1}{3}(2a_m + a_{m-1}) \geqslant \frac{1}{3}(2 \cdot 3 + 3) = 3$$

and

$$a_{m+1} = \frac{1}{3}(2a_m + a_{m-1}) \leqslant \frac{1}{3}(2 \cdot 4 + 4) = 4.$$

Therefore, by the principle of strong mathematical induction it follows that $3 \leqslant a_n \leqslant 4$ for all $n \in \mathbb{N}$. □

Problem 6.3. *Consider the sequence $(a_n)_{n=1}^{\infty}$ recursively defined as $a_1 = 1$, $a_2 = 8$ and for all $n \geqslant 3$, $a_n = a_{n-1} + 2a_{n-2}$. Show that for all $n \geqslant 1$, $a_n = 3 \cdot 2^{n-1} + 2(-1)^n$.*

Solution of Problem 6.3. Note that $3 \cdot 2^{1-1} + 2(-1)^1 = 1 = a_1$, $3 \cdot 2^{2-1} + 2(-1)^2 = 8 = a_2$. Now assume that for some integer $m \geqslant 2$, we have $a_k = 3 \cdot 2^{k-1} + 2(-1)^k$ whenever $1 \leqslant k \leqslant m$. Then we have

$$\begin{aligned} a_{m+1} &= a_m + 2a_{m-1} \\ &= (3 \cdot 2^{m-1} + 2(-1)^m) + 2(3 \cdot 2^{m-2} + 2(-1)^{m-1}) \\ &= 3 \cdot 2 \cdot 2^{m-1} + 2((-1)^m + (-1)^{m-1}) + 2(-1)^{m-1} \\ &= 3 \cdot 2^m + 2(-1)^{m+1}. \end{aligned}$$

Therefore, by the principle of strong mathematical induction it follows that $a_n = 3 \cdot 2^{n-1} + 2(-1)^n$ for all integers $n \geqslant 1$. □

Problem 6.4. *Consider the sequence $(a_n)_{n=1}^{\infty}$ recursively defined as $a_1 = 2$, $a_2 = 4$ and for all $n \geqslant 3$, $a_n = 3a_{n-1} - 2a_{n-2}$. For all $n \geqslant 1$, find a closed formula for a_n.*

Solution of Problem 6.4. After performing a few calculations, it should be clear that $a_n = 2^n$. We prove by induction that this is the correct formula.

Clearly $2^1 = 2 = a_1$ and $2^2 = 4 = a_2$. Now suppose that for some $m \in \mathbb{N}$ with $m \geqslant 2$, we have $a_k = 2^k$ whenever $1 \leqslant k \leqslant m$. Then we have

$$a_{m+1} = 3a_m - 2a_{m-2} = 3 \cdot 2^m - 2 \cdot 2^{m-1} = 3 \cdot 2^m - 2^m = 2^{m+1}.$$

Thus, by the principle of strong mathematical induction it follows that $a_n = 2^n$ for all $n \in \mathbb{N}$. □

7 Set Theory

7.1 Subsets

Problem 7.1. *Prove that $X \subseteq Y$ where $X = \{n \in \mathbb{Z} \mid n \text{ is a multiple of } 6\}$ and $Y = \{n \in \mathbb{Z} \mid n \text{ is even}\}$.*

Solution of Problem 7.1. Let $n \in X$ be given. Then n is a multiple of 6, so there is some $k \in \mathbb{Z}$ such that $n = 6k$. Thus $n = 2(3k)$, so n is even and therefore $x \in Y$. □

Problem 7.2. *Consider the sets*

$$\begin{aligned} A &= \{n \in \mathbb{Z} \mid (\exists k \in \mathbb{Z})(n = 12k + 11)\}, \\ B &= \{n \in \mathbb{Z} \mid (\exists j \in \mathbb{Z})(n = 4j + 3)\}. \end{aligned}$$

(a) *Is $A \subseteq B$? Prove or disprove.*

(b) *Is $B \subseteq A$? Prove or disprove.*

Solution of Problem 7.2. (a) We will show that $A \subseteq B$. Suppose $n \in A$. Then there is some $k \in \mathbb{Z}$ such that $n = 12k + 11$. Put $j = 3k + 2$. Then $j \in \mathbb{Z}$ and

$$n = 12k + 11 = 4(3k) + 4(2) + 3 = 4j + 3,$$

and therefore $n \in B$.

(b) We will show that $B \not\subseteq A$, that is, there is some $n \in B$ such that $n \notin A$. Indeed, put $n = 7$. Then $n = 4(1) + 3$ so $n \in B$. Now assume (towards a contradiction) that $n \in A$. Then there is some $k \in \mathbb{Z}$ such that $7 = 12k + 11$. But then $-4 = 12k$, which is impossible. Thus $n \notin A$ and therefore $B \not\subseteq A$.

□

Problem 7.3. *Consider the sets*

$$A = \{n \in \mathbb{Z} \mid (\exists k \in \mathbb{Z})(n = 4k + 1)\},$$
$$B = \{n \in \mathbb{Z} \mid (\exists j \in \mathbb{Z})(n = 4j - 7\}.$$

Prove that $A = B$.

Solution of Problem 7.3. Suppose $n \in A$. Then there is some $k \in \mathbb{Z}$ such that $n = 4k + 1$. Hence we have

$$n = 4k + 1 = 4(k + 2) - 7 \in B,$$

and therefore $A \subseteq B$.

Conversely, assume $n \in B$. Then there is some $j \in \mathbb{Z}$ such that $n = 4j - 7$. Hence we have

$$n = 4j - 7 = 4(j - 2) + 1 \in A$$

and therefore $B \subseteq A$. □

Problem 7.4. *Consider the sets*

$$A = \{n \in \mathbb{Z} \mid (\exists k \in \mathbb{Z})(n = 3k)\},$$
$$B = \{n \in \mathbb{Z} \mid (\exists i, j \in \mathbb{Z})(n = 15i + 12j)\}.$$

Prove that $A = B$.

Solution of Problem 7.4. Suppose $n \in A$. Then there is some $k \in \mathbb{Z}$ such that $n = 3k$. Note that $k = 5k - 4k$, so we have

$$n = 3k = 3(5k - 4k) = 15(k) + 12(-k) \in B$$

and therefore $A \subseteq B$.

Conversely, suppose $n \in B$. Then there exist $i, j \in \mathbb{Z}$ such that $n = 15i + 12j$. Then $n = 3(5i + 4j) \in A$, and therefore $B \subseteq A$. □

Problem 7.5. *Prove that* $X = \{n \in \mathbb{Z} \mid n + 5 \text{ is odd}\}$ *is the set of all even integers.*

Solution of Problem 7.5. Let $n \in X$ be given. Then $n + 5$ is odd, so there is some $k \in \mathbb{Z}$ such that $n + 5 = 2k + 1$. Thus $n = 2k - 4 = 2(k - 2)$, so n is even.

Now suppose $n \in \mathbb{Z}$ is even. Then there is some $k \in \mathbb{Z}$ such that $n = 2k$. Thus $n + 5 = 2k + 5 = 2(k + 2) + 1$, so $n + 5$ is odd and therefore $n \in X$. □

7.2 Complements

Problem 7.6. *Let A and B be subsets of an ambient set U. Prove that* $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$.

Solution of Problem 7.6. Using the identities found in section Section 4.2, this can be done with a few manipulations:

$$\begin{aligned}
(A - B) \cup (B - A) &= (A \cap \overline{B}) \cup (B \cap \overline{A}) \\
&= (A \cup (B \cap \overline{A})) \cap (\overline{B} \cup (B \cap \overline{A})) \\
&= [(A \cup B) \cap (A \cup \overline{A})] \cap [(\overline{B} \cup B) \cap (\overline{B} \cup \overline{A})] \\
&= (A \cup B) \cap (\overline{B} \cup \overline{A}) \\
&= (A \cup B) \cap \overline{(A \cap B)} \\
&= (A \cup B) - (A \cap B).
\end{aligned}$$

You could also give a a double-inclusion proof.

First, we show that $(A - B) \cup (B - A) \subseteq (A \cup B) - (A \cap B)$. Note that $A \subseteq A \cup B$ and $A \cap B \subseteq B$, so $(A - B) \subseteq (A \cup B) - (A \cap B)$. Similarly, since $B \subseteq A \cup B$ and $A \cap B \subseteq A$, we have $(B - A) \subseteq (A \cup B) - (A \cap B)$, and therefore $(A - B) \cup (B - A) \subseteq (A \cup B) - (A \cap B)$.

Conversely, assume that $x \in (A \cup B) - (A \cap B)$. Then in particular, $x \notin A \cap B$ and $x \in \overline{(A \cap B)}$. If $x \in A$, then $x \notin B$, for otherwise $x \in A \cap B$ and we obtain a contradiction. Thus $x \in A - B \subseteq (A - B) \cup (B - A)$. If now $x \in B$, then $x \notin A$, for otherwise $x \in A \cap B$ and we obtain a contradiction. Thus $x \in B - A \subseteq (A - B) \cup (B - A)$. Therefore, we have $(A \cup B) - (A \cap B) \subseteq (A \cup B) - (A \cap B)$ and equality holds. □

7.3 Arbitrary unions and intersections

Problem 7.7. *For $i \in \mathbb{N}$, let $A_i = (-i, i)$. Compute $\bigcup_{i=1}^{\infty} A_i$.*

Solution of Problem 7.7. We will show that $\bigcup_{i=1}^{\infty} A_i = \mathbb{R}$

- $\bigcup_{i=1}^{\infty} A_i \subseteq \mathbb{R}$

  If $x \in \bigcup_{i=1}^{\infty} A_i$, then $x \in A_{i_0}$ for some i_0. Since $A_{i_0} \subseteq \mathbb{R}$, then $x \in \mathbb{R}$. Since $x \in \bigcup_{i=1}^{\infty} A_i$ was arbitrary, it follows that $\bigcup_{i=1}^{\infty} A_i \subseteq \mathbb{R}$.

- $\mathbb{R} \subseteq \bigcup_{i=1}^{\infty} A_i$

  If $x \in \mathbb{R}$, then there is some $i \in \mathbb{N}$ such that $-i < x < i$. Indeed if $x = 0$ any $i \geqslant 1$ would work, otherwise if $x \neq 0$ then $|x| > 0$ and by the Archimedean principle there is an integer $i \geqslant 1$ such that $|x| < i$ and then $-i < x < i$. Thus $x \in (-i, i) = A_i$ and $x \in \bigcup_{i=1}^{\infty} A_i$. Therefore $\mathbb{R} \subseteq \bigcup_{i=1}^{\infty} A_i$

By combining the two inclusions one has that $\mathbb{R} = \bigcup_{i=1}^{\infty} A_i$. □

Problem 7.8. *For $i \in \mathbb{N}$, let $A_i = (-i, i)$. Compute $\bigcap_{i=1}^{\infty} A_i$.*

Solution of Problem 7.8. We will show that $\bigcap_{i=1}^{\infty} A_i = (-1, 1)$.

- $\bigcap_{i=1}^{\infty} A_i \subseteq (-1, 1)$.

  If $x \notin (-1, 1)$, then $x \notin A_1$, and thus $x \notin \bigcap_{i=1}^{\infty} A_i$. Therefore $\bigcap_{i=1}^{\infty} A_i \subseteq (-1, 1)$.

- $(-1, 1) \subseteq \bigcap_{i=1}^{\infty} A_i$.

  If $x \in (-1, 1)$, then $x \in (-i, i)$ for all $i \in \mathbb{N}$ and thus $(-1, 1) \subseteq \bigcap_{i=1}^{\infty} A_i$.

□

Problem 7.9. *For $i \in \mathbb{N}$, let $A_i = \left[0, 1 - \frac{1}{i}\right]$. Compute $\bigcup_{i \in \mathbb{N}} A_i$.*

Solution of Problem 7.9. We will show that $\bigcup_{i \in \mathbb{N}} A_i = [0, 1)$.

- $\bigcup_{i \in \mathbb{N}} A_i \subseteq [0, 1)$

  If $x \in \bigcup_{i \in \mathbb{N}} A_i$, then $x \in A_j$ for some $j \in \mathbb{N}$. Thus $0 \leqslant x \leqslant 1 - \frac{1}{j} < 1$, and hence $x \in [0, 1)$. Therefore, $\bigcup_{i \in \mathbb{N}} A_i \subseteq [0, 1)$.

- $[0,1) \subseteq \bigcup_{i\in\mathbb{N}} A_i$.

  If now $x \in [0,1)$, then $0 \leqslant x < 1$. Thus $0 < 1-x$ and $\frac{1}{1-x} > 0$, and by the Archimedean principle there is some $j \in \mathbb{N}$ such that $\frac{1}{1-x} \leqslant j$, and hence $\frac{1}{j} \leqslant 1-x$. Thus $0 \leqslant x \leqslant 1-\frac{1}{j}$, and $x \in [0, 1-\frac{1}{j}] = A_j$. Therefore $x \in \bigcup_{i\in\mathbb{N}} A_i$, and $[0,1) \subseteq \bigcup_{i\in\mathbb{N}} A_i$.

By definition of equality between sets we have proven that $\bigcup_{i\in\mathbb{N}} A_i = [0,1)$. □

Problem 7.10. *For $i \in \mathbb{N}$, let $A_i = \left[0, 1-\frac{1}{i}\right]$. Compute $\bigcap_{i\in\mathbb{N}} A_i$.*

Solution of Problem 7.10. We will show that $\bigcap_{i\in\mathbb{N}} A_i = \{0\}$.

- $\bigcap_{i\in\mathbb{N}} A_i \subseteq \{0\}$.

  Suppose $x \in \bigcap_{i\in\mathbb{N}} A_i$. Then in particular $x \in A_1 = [0, 1-1] = \{0\}$, and therefore $\bigcap_{i\in\mathbb{N}} A_i \subseteq \{0\}$.

- $\{0\} \subseteq \bigcap_{i\in\mathbb{N}} A_i$

  Conversely, $0 \in \left[0, 1-\frac{1}{j}\right) = A_j$ for all $j \in \mathbb{N}$, and therefore $\{0\} \subseteq \bigcap_{i\in\mathbb{N}} A_i$.

□

Problem 7.11. *Let $X_n = (\frac{2}{n}, 2n]$ for every integer $n \geqslant 2$.*

1. *Compute $\bigcup_{n=2}^{\infty} X_n$.*
2. *Compute $\bigcap_{n=2}^{\infty} X_n$.*

Solution of Problem 7.11.

1. We will show that $\bigcup_{n=2}^{\infty} X_n = (0,\infty)$.

   First, we show that $\bigcup_{n=1}^{\infty} X_n \subseteq (0,\infty)$. Let $x \in \bigcup_{n=2}^{\infty} X_n$, then there exists $k \geqslant 2$ such that $x \in X_k = (\frac{2}{k}, 2k]$ and hence $\frac{2}{k} < x \leqslant 2k$. Since it follows from $k \geqslant 2$ that $\frac{2}{k} \geqslant 1 > 0$ and $2k < \infty$ one has $0 < x < \infty$ and thus $x \in (0,\infty)$. Therefore $\bigcup_{n=2}^{\infty} X_n \subseteq (0,\infty)$

   Assume now that $x \in (0,\infty)$, then $x > 0$ and also $\frac{x}{2} > 0$. On one hand, if follows from the Archimedean principle that there is some $n_1 \geqslant 2$ such that $n_1 > \frac{x}{2}$, so $2n_1 \geqslant x$. On the other hand, $\frac{2}{x} > 0$ and it follows from the Archimedean principle that there exists $n_2 \geqslant 2$ such that $\frac{2}{x} < n_2$ and hence $x > \frac{2}{n_2}$. Let $k = \max\{n_1, n_2\} \geqslant 2$ then $\frac{2}{k} \leqslant \frac{2}{n_2} < x \leqslant 2n_1 \leqslant k$ and hence $x \in X_k$. Therefore, $(0,\infty) \subseteq \bigcup_{n=2}^{\infty} X_n$.

2. We will show that $\bigcap_{n=2}^{\infty} X_n = (1,4]$.

   Let $x \in \bigcap_{n=2}^{\infty} X_n$ then $x \in X_n = (\frac{2}{n}, 2n]$ for all integers $n \geqslant 2$. In particular, $x \in X_2 = (\frac{2}{2}, 2\cdot 2] = (1,4]$. Therefore, $\bigcap_{n=2}^{\infty} X_n \subseteq (1,4]$.

Now, let $x \in (1,4]$ then $1 < x \leqslant 4$ and for all $n \geqslant 2$, it follows that $\frac{2}{n} \leqslant 1 < x \leqslant 4 \leqslant 2n$. Therefore $x \in (\frac{2}{n}, 2n] = X_n$ for all $n \geqslant 2$, and $(1,4] \subseteq \bigcap_{n=2}^{\infty} X_n$.

□

Problem 7.12. *Let I be a nonempty set and let $\{A_i : i \in I\}$ be an indexed family of sets. Let X be a non-empty set. Suppose that for all $i \in I$, $X \subseteq A_i$. Prove that $X \subseteq \bigcap_{i\in I} A_i$.*

Solution of Problem 7.12. Suppose $x \in X$, and let $i \in I$ be given. Since $X \subseteq A_i$, $x \in A_i$. Since $i \in I$ was arbitrary, $x \in A_i$ for all i, thus $x \in \bigcap_{i\in I} A_i$, and therefore $X \subseteq \bigcap_{i\in I} A_i$. □

Problem 7.13. *Let $\{A_i : i \in \mathbb{N}\}$ be an indexed family of sets. Assume that for all $i \in \mathbb{N}$, $A_{i+1} \subseteq A_i$. Prove that $\bigcup_{i\in\mathbb{N}} A_i = A_1$.*

Solution of Problem 7.13. If $x \in A_1$, then there is some $i \in \mathbb{N}$ such that $x \in A_i$ (namely, $i = 1$), and thus $x \in \bigcup_{i\in\mathbb{N}} A_i$.

Next, we show that $A_i \subseteq A_1$ for all $i \in \mathbb{N}$. Clearly $A_1 \subseteq A_1$. If now $A_m \subseteq A_1$, then $A_{m+1} \subseteq A_m \subseteq A_1$, so $A_{m+1} \subseteq A_1$. Thus it follows by induction that $A_n \subseteq A_1$ for all $n \in \mathbb{N}$.

Now assume $x \in \bigcup_{i\in\mathbb{N}} A_i$. Then there is some $n \in \mathbb{N}$ such that $x \in A_n$. Since $A_n \subseteq A_1$, it follows that $x \in A_1$. □

Problem 7.14. *Let $(X_i)_{i\in I}$ be a collection of subsets of an ambient set U. Show that*

$$\overline{\bigcap_{i\in I} X_i} = \bigcup_{i\in I} \overline{X}_i.$$

Solution of Problem 7.14. We first prove the inclusion $\overline{\bigcap_{i\in I} X_i} \subseteq \bigcup_{i\in I} \overline{X}_i$.

If $\overline{\bigcap_{i\in I} X_i} = \emptyset$ then the inclusion holds, otherwise let $z \in \overline{\bigcap_{i\in I} X_i}$. Then $z \notin \bigcap_{i\in I} X_i$ (by definition of the complement), and it follows that $z \notin X_{i_0}$ for some $i_0 \in I$ (by definition of the intersection). Thus, $z \in \overline{X_{i_0}}$ (by definition of the complement), which means that $z \in \bigcup_{i\in I} \overline{X}_i$ (by definition of the union).

For the reverse inclusion, if $\bigcup_{i\in I} \overline{X}_i = \emptyset$ then the inclusion holds, otherwise let $z \in \bigcup_{i\in I} \overline{X}_i$. Then $z \in \overline{X_{i_0}}$ for some $i_0 \in I$ (by definition of the union), and thus $z \notin X_{i_0}$ (by definition of the complement). It follows that $z \notin \bigcap_{i\in I} X_i$ (by definition of the intersection), and hence $z \in \overline{\bigcap_{i\in I} X_i}$ (by definition of the complement).

Therefore, it follows from the definition of equality between sets that $\overline{\bigcup_{i\in I} X_i} = \bigcap_{i\in I} \overline{X}_i$. □

Problem 7.15. *Let $(X_i)_{i\in I}$ be a collection of subsets of an ambient set U. Show that*

$$\overline{\bigcup_{i\in I} X_i} = \bigcap_{i\in I} \overline{X}_i.$$

Solution of Problem 7.15. We first prove the inclusion $\overline{\bigcup_{i\in I} X_i} \subseteq \bigcap_{i\in I} \overline{X}_i$.

If $\overline{\bigcup_{i\in I} X_i} = \emptyset$ then the inclusion holds, otherwise let $z \in \overline{\bigcup_{i\in I} X_i}$. Then $z \notin \bigcup_{i\in I} X_i$ (by definition of the complement), and it follows that $z \notin X_i$ for all $i \in I$ (by definition of the union). Thus, $z \in \overline{X_i}$ for all $i \in I$ (by definition of the complement), which means that $z \in \bigcap_{i\in I} \overline{X}_i$ (by definition of the intersection).

For the reverse inclusion, if $\bigcap_{i\in I} \overline{X}_i = \emptyset$ then the inclusion holds, otherwise let $z \in \bigcap_{i\in I} \overline{X}_i$. Then $z \in \overline{X_i}$ for all $i \in I$ (by definition of the intersection), and thus $z \notin X_i$ for all $i \in I$ (by definition of the complement). It follows that $z \notin \bigcup_{i\in I} X_i$ (by definition of the union), and hence $z \in \overline{\bigcup_{i\in I} X_i}$ (by definition of the complement).

Therefore, it follows from the definition of equality between sets that $\overline{\bigcup_{i\in I} X_i} = \bigcap_{i\in I} \overline{X}_i$. □

7.4 More problems

Problem 7.16. *Let* $A = \{x + y\sqrt{2} \mid x, y \in Q\} \subseteq \mathbb{R}$.

(a) Prove that for all $x, y \in Q$, $x + y\sqrt{2} = 0$ *if and only if* $x = y = 0$.

(b) Prove that for all $z_1, z_2 \in A$, $z_1 + z_2, z_1 z_2 \in A$ *and , for* $z_2 \neq 0$, $\frac{z_1}{z_2} \in A$.

Solution of Problem 7.16. (a) Clearly if $x = y = 0$ then $x + y\sqrt{2} = 0$. Conversely, assume $x + y\sqrt{2} = 0$. Then $x = -y\sqrt{2}$. If $y \neq 0$ then $\sqrt{2} = -\frac{x}{y} \in \mathbb{Q}$, a contradiction. Thus $y = 0$, and therefore $0 = x + y\sqrt{2} = x$.

(b) Suppose $z_1, z_2 \in A$. Then there exist $x_1, x_2, y_1, y_2 \in \mathbb{Q}$ such that $z_i = x_i + y_i\sqrt{2}$ for $i = 1, 2$. Then we have

$$z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)\sqrt{2} \in A,$$

and

$$z_1 z_2 = (x_1 x_2 + 2y_1 y_2) + (x_1 y_2 + x_2 y_1)\sqrt{2} \in A.$$

Now assume $z_2 \neq 0$. Then we have

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{x_1 + y_1\sqrt{2}}{x_2 + y_2\sqrt{2}} \\ &= \frac{x_1 + y_1\sqrt{2}}{x_2 + y_2\sqrt{2}} \cdot \frac{x_2 - y_2\sqrt{2}}{x_2 - y_2\sqrt{2}} \\ &= \frac{x_1 x_2 - 4y_1 y_2}{x_2^2 + 4y_2^2} + \frac{x_2 y_1 - x_1 y_2}{x_2^2 + 4y_2^2}\sqrt{2}.\end{aligned}$$

Since $\frac{x_1 x_2 - 4y_1 y_2}{x_2^2 + 4y_2^2}, \frac{x_2 y_1 - x_1 y_2}{x_2^2 + 4y_2^2} \in \mathbb{Q}$, it follows that $\frac{z_1}{z_2} \in A$.

□

Problem 7.17. *We say that the sequence of sets* $(X_n)_{n=1}^{\infty}$ *is increasing, or an ascending chain, if* $X_1 \subseteq X_2 \subseteq X_3 \subseteq \cdots \subseteq X_n \subseteq X_{n+1} \subseteq \ldots$. *Formally,* $(X_n)_{n=1}^{\infty}$ *is increasing if*

$$(\forall n \in \mathbb{N})[X_n \subseteq X_{n+1}].$$

Show that the sequence of sets $(X_n)_{n=1}^{\infty}$ *is increasing if and only if*

$$(\forall n \in \mathbb{N})(\forall k \in \mathbb{N})[(n \leqslant k) \implies (X_n \subseteq X_k)].$$

Hint. Your goal is to show that

$$(\forall r \in \mathbb{N})[X_r \subseteq X_{r+1}] \iff (\forall k \in \mathbb{N})(\forall n \in \mathbb{N})[(1 \leqslant n \leqslant k) \implies (X_n \subseteq X_k)]$$

For the implication $\Longleftarrow$ simply put $n = r$ and $k = r+1$. For the implication $\implies$ you need to assume that $(\forall r \in \mathbb{N})[X_r \subseteq X_{r+1}]$ and show by induction on k that $\forall k \in \mathbb{N}\ P(k)$ is true, where $P(k)$ is the predicate $(\forall n \in \mathbb{N})[(1 \leqslant n \leqslant k) \implies (X_n \subseteq X_k)]$. For the base case observe that $P(1)$ is simply $X_1 \subseteq X_1$. For the inductive step $P(k+1)$ is the predicate $(\forall n \in \mathbb{N})[(1 \leqslant n \leqslant k+1) \implies (X_n \subseteq X_{k+1})]$ and you need to distinguish two cases: either $1 \leqslant n \leqslant k$ and you can use the induction hypothesis together with the other assumption, or $n = k+1$ □

Problem 7.18. *We say that the sequence of sets* $(X_n)_{n=1}^{\infty}$ *is decreasing, or a descending chain, if* $X_1 \supseteq X_2 \supseteq X_3 \supseteq \cdots \supseteq X_n \supseteq X_{n+1} \supseteq \ldots$. *Formally,* $(X_n)_{n=1}^{\infty}$ *is increasing if*

$$(\forall n \in \mathbb{N})[X_n \subseteq X_{n+1}].$$

Show that the sequence of sets $(X_n)_{n=1}^{\infty}$ *is decreasing if and only if for all* $n, k \in \mathbb{N}$ *if* $n \leqslant k$ *then* $X_n \supseteq X_k$.

Hint. Your goal is to show that

$$(\forall r \in \mathbb{N})[X_r \supseteq X_{r+1}] \iff (\forall k \in \mathbb{N})(\forall n \in \mathbb{N})[(1 \leqslant n \leqslant k) \implies (X_n \supseteq X_k)]$$

For the implication $\Longleftarrow$ simply put $n = r$ and $k = r+1$. For the implication $\implies$ you need to assume that $(\forall r \in \mathbb{N})[X_r \supseteq X_{r+1}]$ and show by induction on k that $\forall k \in \mathbb{N}\ P(k)$ is true, where $P(k)$ is the predicate $(\forall n \in \mathbb{N})[(1 \leqslant n \leqslant k) \implies (X_n \supseteq X_k)]$. For the base case observe that $P(1)$ is simply $X_1 \supseteq X_1$. For the inductive step $P(k+1)$ is the predicate $(\forall n \in \mathbb{N})[(1 \leqslant n \leqslant k+1) \implies (X_n \supseteq X_{k+1})]$ and you need to distinguish two cases: either $1 \leqslant n \leqslant k$ and you can use the induction hypothesis together with the other assumption, or $n = k+1$ □

Problem 7.19. *Let X and Y be subsets of a universal set U. Show that $\overline{X \cap Y} = \overline{X} \cup \overline{Y}$.*

Solution of Problem 7.19. We first prove the inclusion $\overline{X \cap Y} \subseteq \overline{X} \cup \overline{Y}$. If $\overline{X \cap Y} = \emptyset$ then the inclusion holds, otherwise let $z \in \overline{X \cap Y}$. Then $z \notin X \cap Y$ (by definition of the complement), and it follows that $z \notin X$ or $z \notin Y$ (by definition of the intersection). Thus, $z \in \overline{X}$ or $z \in \overline{Y}$ (by definition of the complement), which means that $z \in \overline{X} \cup \overline{Y}$ (by definition of the union). We just proved that $\overline{X \cap Y} \subseteq \overline{X} \cup \overline{Y}$.

For the reverse inclusion, if $\overline{X} \cup \overline{Y} = \emptyset$ then the inclusion holds, otherwise let $z \in \overline{X} \cup \overline{Y}$. Then $z \in \overline{X}$ or $z \in \overline{Y}$ (by definition of the union), and thus $z \notin X$ or $z \notin Y$ (by definition of the complement). It follows that $z \notin X \cap Y$ (by definition of the intersection), and hence $z \in \overline{X \cap Y}$ (by definition of the complement). This shows the reverse inclusion. □

8 Functions

8.1 Composition

Problem 8.1. *Let $f, g : \mathbb{R} \to \mathbb{R}$ be defined for all $x \in \mathbb{R}$ as $f(x) = x^2 - 3x$ and $g(x) = 5x - 2$.*

1. *Is it possible to define $f \circ g$? If it is, what is $f \circ g$.*
2. *Is it possible to define $g \circ f$? If it is, what is $g \circ f$.*
3. *Are $f \circ g$ and $g \circ f$ equal? (Justify your answer)*

Solution of Problem 8.1.

1. It is possible to define $f \circ g \colon \mathbb{R} \to \mathbb{R}$ and for all $x \in \mathbb{R}$

$$f \circ g(x) = f(g(x)) = (g(x))^2 - 3(g(x)) = (5x-2)^2 - 3(5x-2) = 25x^2 - 35x + 10.$$

2. It is possible to define $g \circ f \colon \mathbb{R} \to \mathbb{R}$ and for all $x \in \mathbb{R}$

$$(g \circ f)(x) = g(f(x)) = 5(f(x)) - 2 = 5(x^2 - 3x) - 2 = 5x^2 - 15x - 2.$$

3. Let $x = 0$ then $(g \circ f)(0) = -2 \neq 10 = (f \circ g)(0)$ and thus $f \circ g \neq g \circ f$.

□

Problem 8.2. *Let $f, g : \mathbb{Z} \to \mathbb{Z}$ be defined for all $n \in \mathbb{Z}$ as $f(n) = 2n + 3$ and*

$$g(n) = \begin{cases} 2n - 1 & \text{if } n \text{ is even,} \\ n + 1 & \text{if } n \text{ is odd.} \end{cases}$$

1. *Is it possible to define $f \circ g$? If it is, what is $f \circ g$.*

2. *Is it possible to define $g \circ f$? If it is, what is $g \circ f$.*

3. *Are $f \circ g$ and $g \circ f$ equal? (Justify your answer)*

Solution of Problem 8.2.

1. It is possible to define $f \circ g \colon \mathbb{Z} \to \mathbb{Z}$ and for all $n \in \mathbb{Z}$, since we have $f(2n-1) = 4n+1$ and $f(n+1) = 2n+5$, it follows that

$$(f \circ g)(n) = \begin{cases} 4n+1 & \text{if } n \text{ is even,} \\ 2n+5 & \text{if } n \text{ is odd.} \end{cases}$$

2. Since $f(n)$ is odd for any $n \in \mathbb{Z}$, we have

$$(g \circ f)(n) = f(n) + 1 = (2n+3) + 1 = 2n+4.$$

3. Let $n = 0$ then n is even and $(g \circ f)(0) = 4 \neq 1 = (f \circ g)(0)$. Therefore, $g \circ f \neq f \circ g$.

□

8.2 Injectivity, surjectivity, bijectivity

Problem 8.3. *For $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x + |x|$, determine if:*

1. *f is injective,*

2. *f is surjective,*

3. *f is bijective.*

Solution of Problem 8.3. First, note that

$$f(x) = \begin{cases} 2x & : \quad x \geqslant 0, \\ 0 & : \quad x < 0. \end{cases}$$

1. f is not injective, since $f(-1) = 0 = f(0)$.

2. f is not surjective, since $y \in \mathbb{R}$ is not in the range of f whenever $y < 0$. Indeed, for any $x \in \mathbb{R}$ we have $x + |x| \geqslant x + (-x) = 0$, so $f(x) \geqslant 0$.

3. f is not bijective because it is not injective (nor surjective).

□

8.3 Composition and injectivity/surjectivity

Problem 8.4. *Let W, X, Y be nonempty sets. Let $f : W \to X$, $g : X \to Y$ be functions. Show that if $g \circ f$ is surjective, then g is surjective.*

Solution of Problem 8.4. Fix $y \in Y$. Since $g \circ f$ is surjective, there is some $w \in W$ such that $(g \circ f)(w) = y$. Put $x = f(w)$. Then $x \in X$ and $g(x) = g(f(w)) = (g \circ f)(w) = y$. Therefore, g is surjective. □

Problem 8.5. *Let W, X, Y be nonempty sets. Let $f : W \to X$, $g : X \to Y$ be functions. Show that if $g \circ f$ is injective, then f is injective.*

Solution of Problem 8.5. Assume that $g \circ f$ is injective. Let $w_1, w_2 \in W$ such that $f(w_1) = f(w_2)$. Since g is a function one has $g(f(w_1)) = g(f(w_2))$ and $(g \circ f)(w_1) = (g \circ f)(w_2)$ (by definition of the composition). Since $g \circ f$ is injective it implies that $w_1 = w_2$, and f is injective. □

Problem 8.6. *Let X and Y be nonempty sets and let $f : X \to Y$ be a function. Prove that f is injective if and only if for all sets Z, for all functions $h : Z \to X$ and $k : Z \to X$, if $f \circ h = f \circ k$, then $h = k$.*

Solution of Problem 8.6. First suppose f is injective. Let Z be a set, and let $h, k : Z \to X$ be functions such that $f \circ h = f \circ k$. Given $z \in Z$, since $f(h(z)) = f(k(z))$, it follows that $h(z) = k(z)$. Thus $h = k$.

Conversely assume f is not injective. Then there exist $x_1, x_2 \in X$ such that $x_1 \neq x_2$ while $f(x_1) = f(x_2)$. Define $h, k : X \to X$ by $h(x) = x_1$ and $k(x) = x_2$ for all $x \in X$. Then for all $x \in X$, $f(h(x)) = f(x_1) = f(x_2) = f(k(x))$, so $f \circ h = f \circ k$, but $h \neq k$. □

Problem 8.7. *Let X and Y be nonempty sets and let $f : X \to Y$ be a function. Prove that f is surjective if and only if for all sets Z, for all functions $h : Y \to Z$ and $k : Y \to Z$, if $h \circ f = k \circ f$, then $h = k$.*

Solution of Problem 8.7. First assume f is surjective. Let Z be a set, and let $h, k : Y \to Z$ be functions such that $h \circ f = k \circ f$. Given $y \in Y$, there is some $x \in X$ such that $y = f(x)$, and thus $h(y) = h(f(x)) = k(f(x)) = k(y)$. Therefore, $h = k$.

Conversely, assume f is not surjective. Then there is some $y_0 \in Y$ such that the set $\{x \in X : f(x) = y_0\}$ is empty. Put $h = i_Y$, and let $k : Y \to Y$ be defined by $k(y) = y$ if $y \neq y_0$ and $k(y_0) = y_1$ for some $y_1 \in Y$ with $y_0 \neq y_1$. Then $h \neq k$ since $h(y_0) = y_0 \neq y_1 = k(y_0)$, but $h \circ f = k \circ f$. □

9 Injectivity, surjectivity, and one-sided invertibility

Problem 9.1. *Let X and Y be nonempty sets and $f : X \to Y$ be a function. We say that f is left-invertible (or admits a left-inverse) if there exists a function $g : Y \to X$ such that $g \circ f = i_X$. Prove that f is injective if* *and only if f is left-invertible.*

Solution of Problem 9.1. Suppose first that f is left-invertible. Then there is some function $g: Y \to X$ such that $g \circ f = i_X$. Let $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$, then $g(f(x_1)) = g(f(x_2))$ (since g is a function), and thus $(g \circ f)(x_1) = (g \circ f)(x_2)$ (by definition of the composition). It follows from the assumption that $i_X(x_1) = i_X(x_2)$ and hence $x_1 = x_2$ (by definition of the identity function on X). Therefore f is injective.

Conversely, assume f is injective. Define a function $g: Y \to X$ as follows: if $y = f(x)$ for some (and hence only one by injectivity) $x \in X$, put $g(y) = x$, and otherwise define $g(y)$ arbitrarily. Then for each $x \in X$, let $y = f(x)$. Then $g(y) = x$, that is, $g(f(x)) = x$ and hence $(g \circ f)(x) = x$ (by definition of the composition). By definition of the identity function $(g \circ f)(x) = i_X(x)$ and thus $g \circ f = i_X$. Therefore f is left-invertible. □

Problem 9.2. *Let X and Y be nonempty sets, and $f\colon X \to Y$ be a function. We say that f is right-invertible (or admits a right-inverse) if there exists a function $g\colon Y \to X$ such that $f \circ g = i_Y$. Prove that if f has a right-inverse then f is surjective.*

Solution of Problem 9.2. Suppose first that f is right-invertible. Then there is some function $h: Y \to X$ such that $f \circ h = i_Y$. Let $y \in Y$, then

$$\begin{aligned} y &= i_Y(y) \text{ (by definition of the identity function on } Y) \\ &= (f \circ h)(y) \text{ (since } f \circ h(y) = i_Y(y) \text{ by definition of } h \text{ being a right-inverse of } f) \\ &= f(h(y)) \text{ (by definition of the composition).} \end{aligned}$$

If we let $x = h(y)$ then $x \in X$ (since the codomain of h is X) and $y = f(x)$. We just proved that for all $y \in Y$, there is $x \in X$ such that $y = f(x)$, which means that f is surjective. □

10 Functions and sets

Problem 10.1. *Let X and Y be nonempty sets, and $f\colon X \to Y$ be an injective function. Let A be a subset of X. Prove that $f^{-1}(f(A)) = A$.*

Solution of Problem 10.1. The result is proved by a double inclusion argument. We first prove that $f^{-1}(f(A)) \subseteq A$. If $f^{-1}(f(A)) = \emptyset$ then the inclusion holds. Otherwise let $x \in f^{-1}(f(A))$, then $f(x) \in f(A)$ (by definition of the inverse image of a subset), and there exists $a \in A$ such that $f(x) = f(a)$ (by definition of the image of a subset). Since f is injective it follows that $x = a$, and hence $x \in A$ (because $a \in A$).

We now prove that $A \subseteq f^{-1}(f(A))$. If $A = \emptyset$ the inclusion holds. Otherwise, let $x \in A$, then $f(x) \in f(A)$ (by definition of the image of a subset) and $x \in f^{-1}(f(A))$ (by definition of the inverse image of a subset).

□

Problem 10.2. *Let X and Y be nonempty sets, and $f\colon X \to Y$ be an surjective function. Let A be a subset of Y. Prove that $f(f^{-1}(A)) = A$.*

Solution of Problem 10.2. The result is proved by a double inclusion argument. We first prove that $f(f^{-1}(A)) \subseteq A$. If $f(f^{-1}(A)) = \emptyset$ the inclusion holds. Otherwise let $y \in f(f^{-1}(A))$, then $y = f(x)$ for some $x \in f^{-1}(A)$ (by definition of the image), and $f(x) \in A$ (by definition of the inverse image). But $y = f(x)$ belongs to A since $f(x)$ does. Therefore $f^{-1}(f(A)) \subseteq A$.

We now prove that $A \subseteq f(f^{-1}(A))$. If $A = \emptyset$ the inclusion holds. Otherwise let $a \in A$, then $a \in Y$ since A is a subset of Y. By surjectivity of f, there exists $x \in X$ such that $a = f(x)$, and $f(x) \in A$ (since a is in A). It follows that $x \in f^{-1}(A)$ (by definition of the inverse image) and $f(x) \in f(f^{-1}(A))$ (by definition of the image). Therefore $a \in f(f^{-1}(A))$. □

11 Supplementary problems

Problem 11.1. *Let $f_1\colon X_1 \to X_2$, $f_2\colon X_2 \to X_3$, $f_3\colon X_3 \to X_4$ and $f_4\colon X_4 \to X_5$. Show that $((f_4 \circ f_3) \circ f_2) \circ f_1 = f_4 \circ (f_3 \circ (f_2 \circ f_1))$.*

Problem 11.2. *Let X and Y be nonempty sets, and $f\colon X \to Y$ be a function. Prove that f is surjective then f is right-invertible.*

Solution of Problem 11.3. Assume f is surjective. For each $y \in Y$, the set $\{x \in X : f(x) = y\}$ is non-empty. Note that if $y_1 \neq y_2$, then $\{x \in X : f(x) = y_1\} \cap \{x \in X : f(x) = y_2\} = \varnothing$. By the axiom of choice, there is a function $h : Y \to X$ such that for each $y \in Y$, $h(y) \in \{x \in X : f(x) = y\}$. Hence $f(h(y)) = y$ for each $y \in Y$, so $f \circ h = i_Y$ and therefore f is right-invertible. □

Problem 11.3. *Let $f_1\colon X_1 \to X_2$, $f_2\colon X_2 \to X_3$, $f_3\colon X_3 \to X_4$ be three injective functions. Show that $f_3 \circ f_2 \circ f_1$ is injective.*