REAL ANALYSIS MATH 608
HOMEWORK #3

Problem 1.

(1) *Let $f\colon X \to Y$ be a bijection that is continuous, where X is a compact topological space and Y is a Hausdorff topological space. Show that $f^{-1}\colon Y \to X$ is continuous.*
(2) *Show that a metric space is normal.*

Problem 2. *Let X be a set and $\{f_i\colon X \to (Y_i, \tau_i)\}_{i\in I}$ be a collection of functions. Let τ_{ini} be the initial topology on X generated by $\{f_i\}_{i\in I}$.*

(1) *Show that a net $(x_\alpha)_{\alpha\in D}$ in X converges to $x \in X$ in the initial topology if and only if $(f_i(x_\alpha))_{\alpha\in D}$ converges to $f_i(x)$, for all $i \in I$.*
(2) *Interpret the previous result (restricting to sequences) in the case when for all $i \in I$, $Y_i = Y$, $X = \prod_{i\in I} Y_i = Y^I$ and $f_i = \pi_i$ is the canonical projection on the i-th coordinate.*

Problem 3. *Assume that $(X_i)_{i\in I}$ is a family of topological spaces for which infinitely many are not compact. Let K be a subset of $\prod_{i\in I} X_i$ that is closed and compact in the product topology. Show that K is nowhere dense.*

Problem 4. *Let X be a set and $\{f_i\colon X \to (Y_i, \tau_i)\}_{i\in I}$ be a collection of functions. Let τ_{ini} be the initial topology on X generated by $\{f_i\}_{i\in I}$*

(1) *Show that if (X, τ_{ini}) is a T_0-space, then for all $x \neq y \in X$ there exists $i_0 \in I$ such that $f_{i_0}(x) \neq f_{i_0}(y)$.*
(2) *Let $k \in \{0, 1, 2\}$. Assume that for all $i \in I$ (Y_i, τ_i) is a T_k-space . Show that if for all $x \neq y \in X$ there exists $i_0 \in I$ such that $f_{i_0}(x) \neq f_{i_0}(y)$, then (X, τ_{ini}) is a T_k-space.*

Problem 5. *Let (X, d_X) be a metric space and $\alpha > 0$. We say that a continuous function $f\colon X \to \mathbb{C}$ is* Hölder continuous of exponent α, *if there is $C > 0$ such that for all $x, y \in X$,*

$$|f(x) - f(y)| \leqslant C d_X^\alpha(x, y),$$

or equivalently

$$Lip_\alpha(f) \overset{\text{def}}{=} \sup_{x\neq y} \frac{|f(x) - f(y)|}{d_X^\alpha(x, y)} < \infty.$$

Show that if X is compact then the set

$$S = \{f \in C(X)\colon \sup_{x\in X} |f(x)| \leqslant 1 \text{ and } Lip_\alpha(f) \leqslant 1\}.$$

is compact (for the uniform topology on $C(X)$).