REAL ANALYSIS MATH 608
HOMEWORK #3

Problem 1.

(1) Let $f\colon X \to Y$ be a bijection that is continuous, where X is a compact topological space and Y is a Hausdorff topological space. Show that $f^{-1}\colon Y \to X$ is continuous.
(2) Show that a metric space is normal.

Solution.

(1) Observe that it follows immediately from the identity $g^{-1}(A) = g^{-1}(A^c)^c$, that a function g is continuous of a function if and only if the inverse image of a closed set is closed.
Let $F \subset X$ be closed. Then F is compact since X is compact. Now, $(f^{-1})^{-1}(F) = f(F)$ which is compact since f is continuous. But $f(F)$ is closed in Y since Y is Hausdorff, and hence f^{-1} is continuous by the observation above.

(2) Let F, G be closed and disjoint subsets of (X, d_X). Observe that $d(x,A) \stackrel{\text{def}}{=} \inf\{d_X(x,a)\colon a \in A\} = 0$ if and only if $x \in \overline{A}$. Indeed, if x is in the closure of A then there is a sequence $(a_n)_n$ in A such that $\lim_n d(x,a_n) = 0$ and hence $d(x,A) = 0$. If $d(x,A) = 0$ then by definition there is a sequence $(a_n)_n$ in A such that $\lim_n d(x,a_n) = 0$, which means that x is in the closure of A. Consider the sets $U \stackrel{\text{def}}{=} \cup_{x\in F} B(x, \frac{1}{3}d(x,G))$ and $V \stackrel{\text{def}}{=} \cup_{y\in G} B(y, \frac{1}{3}d(y,F))$. By the observation above, the radii are all positive because F and G are disjoint, and U and V are open sets such that $F \subset U$ and $G \subset V$. It remains to show that they are disjoint. Assume that there is $z \in U \cap V$, then $z \in B(x, \frac{1}{3}d(x,G)) \cap B(y, \frac{1}{3}d(y,G))$ for some $x \in F$ and $y \in G$. The following inequalities provide a contradiction:

$$\max\{d(x,G), d(y,G)\} \leqslant d_X(x,y) \leqslant d_X(x,z) + d_X(z,y) \leqslant \frac{1}{3}d(x,G) + \frac{1}{3}d(y,G) \leqslant \frac{2}{3}\max\{d(x,G), d(y,G)\}.$$

□

Problem 2. *Let X be a set and $\{f_i\colon X \to (Y_i, \tau_i)\}_{i\in I}$ be a collection of functions. Let τ_{ini} be the initial topology on X generated by $\{f_i\}_{i\in I}$.*

(1) Show that a net $(x_\alpha)_{\alpha\in D}$ in X converges to $x \in X$ in the initial topology if and only if $(f_i(x_\alpha))_{\alpha\in D}$ converges to $f_i(x)$, for all $i \in I$.
(2) Interpret the previous result (restricting to sequences) in the case when for all $i \in I$, $Y_i = Y$, $X = \Pi_{i\in I} Y_i = Y^I$ and $f_i = \pi_i$ is the canonical projection on the i-th coordinate.

Solution. (1) One direction is easy since by definition, for all i, $f_i\colon (X, \tau_{ini}) \to (Y_i, \tau_i)$ is continuous and thus $(f_i(x_\alpha))_{\alpha\in D}$ converges to $f_i(x)$ whenever $(x_\alpha)_{\alpha\in D}$ converges to $x \in X$ in the initial topology.
For the other implication, recall that a neighborhood basis of $x \in X$ for the initial topology consists of sets of the form $\cap_{i\in F} f_i^{-1}(U_i)$, where F is a finite subset of I, and $U_i \in \tau_i$. If for all $i \in I$, $(f_i(x_\alpha))_{\alpha\in D}$ converges to $f_i(x)$, then $(f_i(x_\alpha))_{\alpha\in D}$ is eventually in U_i for all $i \in F$, and since F is finite we can certainly find $\beta \in D$ such that for all $\alpha \geqslant \beta$, and all $i \in F$, $f_i(x_\alpha) \in U_i$. Taking inverse images, this means that $(x_\alpha)_{\alpha\in D}$ is eventually in $\cap_{i\in F} f_i^{-1}(U_i)$, from which we conclude that $(x_\alpha)_{\alpha\in D}$ converges to $x \in X$ in the initial topology.

(2) A sequence in Y^I can (should!) be seen as a sequence of functions $\{g_n\colon I \to Y\}_{n\geqslant 1}$. By (1), $(g_n)_n$ converges to $g\colon I \to Y$ in the initial topology generated by the projections (which is the product

topology on Y^I) if and only if for all $i \in I$, $\pi_i(g_n)$ converges to $\pi_i(g)$. Since $\pi_i(g_n) = g_n(i)$, we can rephrase this by saying that $(g_n)_n$ converges to g in the product topology if and only if $(g_n)_n$ converges pointwise to g. This is the reason why the product topology is also called the topology of pointwise convergence.

□

Problem 3. *Assume that $(X_i)_{i\in I}$ is a family of topological spaces for which infinitely many are not compact. Let K be a subset of $\prod_{i\in I} X_i$ that is closed and compact in the product topology. Show that K is nowhere dense.*

Solution. Assume the claim is not true and hence $K^\circ \neq \emptyset$. Thus there is an $x \in K$ and a neighborhood U of x so that $U \subset K$.

We can assume that U is of the form $U = \prod_{i\in I} U_i$ with U_i open in X_i for $i \in I$, and with $F = \{i \in I\colon U_i \neq X_i\}$ is finite.

Since $\{i \in I\colon X_i \text{ not compact}\}$ is infinite, we can choose an $i_0 \in I$ so that X_{i_0} is not compact and so that $U_{i_0} = X_{i_0}$.

Now since continuous images of compact sets are also compact, the projection $\pi_{i_0}(K)$ onto X_{i_0} is compact, and therefore cannot be equal to whole space X_{i_0}. We deduce that there must be an $x_{i_0} \in X_{i_0} \setminus \pi_{i_0}(K) = U_{i_0} \setminus \pi_{i_0}(K)$ and for $i \in I \setminus \{i_0\}$ we pick an $x_i \in U_i$. Therefore $(x_i)_{i\in I} \in U \subset K$, but, since $x_{i_0} \notin \pi_{i_0}(K)$ we have that $(x_i)_{i\in I} \notin K$, which is a contradiction. □

Problem 4. *Let X be a set and $\{f_i\colon X \to (Y_i, \tau_i)\}_{i\in I}$ be a collection of functions. Let τ_{ini} be the initial topology on X generated by $\{f_i\}_{i\in I}$*

(1) Show that if (X, τ_{ini}) is a T_0-space, then for all $x \neq y \in X$ there exists $i_0 \in I$ such that $f_{i_0}(x) \neq f_{i_0}(y)$.
(2) Let $k \in \{0, 1, 2\}$. Assume that for all $i \in I$ (Y_i, τ_i) is a T_k-space . Show that if for all $x \neq y \in X$ there exists $i_0 \in I$ such that $f_{i_0}(x) \neq f_{i_0}(y)$, then (X, τ_{ini}) is a T_k-space.

Solution. (1) Let $x \neq y \in X$, Since τ_{ini} is T_0, there is an open set $U \in \tau_{ini}$ such that, say, $x \in U$ and $y \notin U$. Let F a finite subset of I, and U_i open set in Y_i such that $x \in \cap_{i\in F} f_i^{-1}(U_i) \subset U$. Since $y \notin \cap_{i\in F} f_i^{-1}(U_i)$, there is $i_0 \in F$ such that $f_{i_0}(y) \notin U_{i_0}$. Since $f_{i_0}(x) \in U_{i_0}$ it follows that $f_{i_0}(x) \neq f_{i_0}(y)$.

(2) We will prove it for the T_2 separation property. Let $x \neq y \in X$, then $f_{i_0}(x) \neq f_{i_0}(y)$, and by Haussdorfness of Y_{i_0}, there are disjoint open sets U, V such that $f_{i_0}(x) \in U$ and $f_{i_0}(y) \in V$, i.e. $x \in f_{i_0}^{-1}(U)$ and $y \in f_{i_0}^{-1}(V)$. Since $f_{i_0}^{-1}(U)$ and $f_{i_0}^{-1}(V)$ are open sets for the initial topology and are clearly disjoint, the conclusion follows. A similar argument can be given to treat the two other separation properties. If Y_{i_0} is only T_1, then there is a disjoint open set U such that $f_{i_0}(x) \in U$ and $f_{i_0}(y) \notin U$, i.e. $x \in f_{i_0}^{-1}(U)$ and $y \notin f_{i_0}^{-1}(U)$. Since $f_{i_0}^{-1}(U)$ and $f_{i_0}^{-1}(V)$ are open sets for the initial topology and are clearly disjoint, the conclusion follows observing that the role of x and y can be reversed.

If Y_{i_0} is only T_0, then there is a disjoint open set U such that, say, $f_{i_0}(x) \in U$ and $f_{i_0}(y) \notin U$, i.e. $x \in f_{i_0}^{-1}(U)$ and $y \notin f_{i_0}^{-1}(U)$. Since $f_{i_0}^{-1}(U)$ and $f_{i_0}^{-1}(V)$ are open sets for the initial topology and are clearly disjoint, the conclusion follows.

□

Problem 5. *Let (X, d_X) be a metric space and $\alpha > 0$. We say that a continuous function $f\colon X \to \mathbb{C}$ is* Hölder continuous of exponent α*, if there is $C > 0$ such that for all $x, y \in X$,*

$$|f(x) - f(y)| \leqslant C d_X^\alpha(x, y),$$

or equivalently

$$Lip_\alpha(f) \stackrel{\text{def}}{=} \sup_{x \neq y} \frac{|f(x) - f(y)|}{d_X^\alpha(x, y)} < \infty.$$

Show that if X is compact then the set

$$S = \{f \in C(X)\colon \sup_{x \in X} |f(x)| \leqslant 1 \text{ and } Lip_\alpha(f) \leqslant 1\}.$$

is compact (for the uniform topology on $C(X)$).

Solution. This is an ultra-typical application of Arzelà-Ascoli.

- S **is pointwise bounded:** This is immediate from the condition $\sup_{x \in X} |f(x)| \leqslant 1$ in the definition of S and the condition (it is in fact uniformly bounded)
- S **is equicontinuous:** This follows from the condition $Lip_\alpha(f) \leqslant 1$ in the definition of S (in fact S is equi-uniformly continuous). Let $\varepsilon > 0$ and choose $\delta = \varepsilon^{1/\alpha} > 0$ (which does not depend on any $f \in S$ nor $x \in X$). Then for all $x, y \in X$, and all $f \in S$, if $d_X(x, y) < \delta$ it follows that

$$|f(x) - f(y)| \leqslant d^\alpha(x, y) < \delta^\alpha = \varepsilon,$$

- S **is closed:** This will be true by continuity of the module and because the defining conditions for S involve non-strict inequalities and are "closed" conditions. Assume that $(f_n)_n \subset S$ and $f \in C(X)$ so that $d_u(f_n, f) \to 0$ for $n \to \infty$. Then

$$\begin{aligned}
\sup_{x \neq y} \frac{|f(x) - f(y)|}{d^\alpha(x, y)} &= \sup_{x \neq y} \lim_{n \to \infty} \frac{|f_n(x) - f_n(y)|}{d^\alpha(x, y)} \\
&\leqslant \sup_{x \neq y} \sup_{n \in \mathbb{N}} \frac{|f_n(x) - f_n(y)|}{d^\alpha(x, y)} \\
&= \sup_{n \in \mathbb{N}} \sup_{x \neq y} \frac{|f_n(x) - f_n(y)|}{d^\alpha(x, y)} = \sup_{n \in \mathbb{N}} Lip_\alpha(f_n) \leqslant 1.
\end{aligned}$$

  Moreover, since for all $x \in X$, $|f(x)| = \lim_n |f_n(x)|$ (by continuity of the $|\cdot|$), it follows that $\sup_{x \in x} |f(x)| \leqslant 1$. Thus $f \in S$, and S is closed.

The conclusion now follows from Arzelà-Ascoli theorem.

□