REAL ANALYSIS MATH 608
HOMEWORK #4

Problem 1. *Let (X,d) be a metric space, $S \subset X$.*

1. *Show that if S is compact then S is totally bounded.*
2. *Assume that (X,d) is complete. Show that if S is totally bounded then the closure of S is compact.*

Problem 2. *Let (X,τ) be a topological compact space and (Y,d) be a complete metric space. $C(X,Y) \stackrel{\text{def}}{=} \{f\colon X \to Y \mid f \text{ continuous }\}$ is a complete metric space when equipped wit the uniform metric d_u. Let $\mathcal{F} \subset C(X,Y)$ and assume that the closure of $\mathcal{F}$ is compact.*

1. *Show that $\mathcal{F}$ is equi-continuous.*
2. *Show for every $x \in X$, the closure of $\{f(x)\colon f \in \mathcal{F}\}$ is compact in (Y,d).*

Hint:
(1): Use the fact that $\mathcal{F}$ is totally bounded (and justify it).
(2): Consider the evaluation map $\delta_x\colon f \mapsto f(x)$.

Problem 3. *Show that every compact metric space is homeomorphic to a closed subset of $[0,1]^{\mathbb{N}}$.*

Problem 4. *Assume that $(X_i)_{i\in I}$ is a family of topological spaces for which infinitely many are not compact. Let K be a subset of $\prod_{i\in I} X_i$ that is closed and compact in the product topology. Show that K is nowhere dense.*

Problem 5. *Let $k\colon [0,1]\times[0,1] \to \mathbb{R}$ be continuous. For $f \in C([0,1])$ define $T(f)\colon [0,1] \to \mathbb{R}$ by:*

$$T(f)(x) = \int_0^1 k(x,y)f(y)dy, \ x \in [0,1].$$

1. *Show that $T(C[0,1]) \subset C([0,1])$.*
2. *A bounded set in $C([0,1])$ is a set S for which there exists an $R > 0$ so that $d_u(0,f) \leqslant R$ for all $f \in S$. Show that T maps bounded sets into compact set.*