REAL ANALYSIS MATH 608
HOMEWORK #4

Problem 1. *Let (X,d) be a metric space, $S \subset X$.*

(1) Show that if S is compact then S is totally bounded.
(2) Assume that (X,d) is complete. Show that if S is totally bounded then the closure of S is compact.

Solution. (1) Let $\varepsilon > 0$ and observe that $\cup_{x\in S} B(x,\varepsilon)$ is an open covering of S since open balls are open. By compactness of S, we can extract a finite subcover $\cup_{i=1}^{k} B(x_i,\varepsilon)$, and we are done.

(2) One implication was taken care off in (1). For the other implication, assume that S is totally bounded. Observe first that $\overline{S}$ is also totally bounded. Indeed, if $\cup_{i=1}^{k} B(x_i,\varepsilon)$ covers S, then $\cup_{i=1}^{k} \overline{B}(x_i,\varepsilon)$ covers S and is closed ($\overline{B}(x,r)$ is the closed ball of radius r centered at x), and hence $\overline{S}$ is covered by $\cup_{i=1}^{k} \overline{B}(x_i,\varepsilon)$. Therefore, $\cup_{i=1}^{k} B(x_i,2\varepsilon)$ is a cover of $\overline{S}$ by open balls (then adjust the epsilon's).

Now, consider a sequence $\{x_n\}_{n=1}^{\infty}$ in $\overline{S}$ and let $\cup_{i=1}^{k} \overline{B}(s_i^{(1)},1)$ be a cover of $\overline{S}$. WLOG we can assume that the $s_i^{(1)}$'s are in $\overline{S}$ (do you see why?), and there is a point $y_1 \in S$ (it is one of the $s_i^{(1)}$'s) such that there is an infinite subsequence $(x_n^{(1)})_n$ in $B(y_1,1)$. Covering $B(y_1,1)$ (which is also totally bounded) with $\cup_{i=1}^{k} \overline{B}(s_i^{(2)},\frac{1}{2})$, there is a point $y_2 \in B(y_1,1)$ such that there is an infinite subsequence $(x_n^{(2)})_n$ of $(x_n^{(1)})_n$ in $B(y_2,\frac{1}{2})$. Continuing the recursive construction, for all $k \geqslant 1$ there are $y_{k+1} \in B(y_k,\frac{1}{k})$ and an infinite subsequence $(x_n^{(k+1)})_n$ of $(x_n^{(k)})_n$ in $B(y_{k+1},\frac{1}{k+1})$. Since $d(x_n^{(n)},x_k^{(k)}) \leqslant d(x_n^{(n)},y_n) + d(y_n,y_k) + d(y_k,x_k^{(k)}) \leqslant \frac{1}{n} + \frac{1}{\max\{n,k\}} + \frac{1}{k}$, the diagonal sequence $(z_n)_n = (x_n^{(n)})_n$ is a Cauchy sequence in $\overline{S}$ which converges by completness to $z \in \overline{S}$. We just show that $\overline{S}$ is sequentially compact and hence compact since we are in the metric space setting.

□

Problem 2. *Let (X,τ) be a topological compact space and (Y,d_Y) be a complete metric space. $C(X,Y) \overset{\text{def}}{=} \{f\colon X \to Y \mid f \text{ continuous }\}$ is a complete metric space when equipped wit the uniform metric d_u. Let $\mathcal{F} \subset C(X,Y)$ and assume that the closure of $\mathcal{F}$ is compact.*

(1) Show that $\mathcal{F}$ is equi-continuous.
(2) Show for every $x \in X$, the closure of $\{f(x)\colon f \in \mathcal{F}\}$ is compact in (Y,d_Y).

Hint:
(1): Use the fact that $\mathcal{F}$ is totally bounded (and justify it).
(2): Consider the evaluation map $\delta_x\colon f \mapsto f(x)$.

Solution. (1) Since $C(X,Y)$ is complete, and $\overline{\mathcal{F}}$ is compact, it follows from Problem 1 that $\overline{\mathcal{F}}$, and in turn $\mathcal{F}$, is totally bounded. So cover $\mathcal{F}$ by finitely many balls of radius ε, say, $\cup_{i=1}^{k} B_{d_u}(f_i,\varepsilon)$. The rest of the argument is a typically $3-\varepsilon$ type argument. Let $x_0 \in X$, and $f \in \mathcal{F}$ then there is $1 \leqslant i_0 \leqslant k$ such that $\sup_{x\in X} d_Y(f_{i_0}(x),f(x)) < \varepsilon$. Then,

$$d_Y(f(y),f(x_0)) \leqslant d_Y(f(y),f_{i_0}(y)) + d_Y(f_{i_0}(y),f_{i_0}(x_0)) + d_Y(f_{i_0}(x_0),f(x_0)) \leqslant 2\varepsilon + d_Y(f_{i_0}(y),f_{i_0}(x_0)).$$

Now, for all $1 \leqslant i \leqslant k$, by continuity of f_i there is an open set U_i that contains x_0 such that $y \in U_i \implies d_Y(f_i(y),f_i(x_0)) \leqslant \varepsilon$. Since i_0 depends on f we need to consider the set $\cap_i^k U_i$ which is open and contains x. Then for all $y \in \cap_i^k U_i$, one has $d_Y(f(y), f(x_0)) \leqslant 2\varepsilon + d_Y(f_{i_0}(y), f_{i_0}(x_0)) \leqslant 3\varepsilon$, which proves equicontinuity.

(2) Consider the evaluation map $\delta_x \colon f \mapsto f(x)$. Then δ_x is a 1-Lipschitz maps (since $d_Y(\delta_x(f), \delta_x(g)) = d_Y(f(x), g(x)) \leqslant \sup_{x\in X} d_Y(f(x), g(x)) = d_u(f,g)$) and thus is continuous. Now, observe that $\{f(x) \colon f \in \mathcal{F}\} = \delta_x(\mathcal{F})$, and hence $\overline{\{f(x) \colon f \in \mathcal{F}\}} = \overline{\delta_x(\mathcal{F})} \subset \overline{\delta_x(\overline{\mathcal{F}})}$. Since $\delta_x(\overline{\mathcal{F}})$ is compact as the image of a compact by a continuous map, it is also closed (it is a compact in a Hausdorff space), and hence $\overline{\{f(x) \colon f \in \mathcal{F}\}} \subset \delta_x(\overline{\mathcal{F}})$ is a closed subset of a compact space, and it is compact.

□

Problem 3. *Show that every compact metric space is homeomorphic to a closed subset of* $[0,1]^{\mathbb{N}}$.

Solution. Every compact metric space (K,d) is separable. So let $\{x_n\}_{n=1}^{\infty}$ be a dense sequence in X. A compact metric space is bounded and let $D = \sup_{x \neq y \in K} d(x,y) < \infty$. Now, consider the embedding $f \colon K \to [0,1]^{\mathbb{N}}$ defined by $f(x) = (\frac{1}{D} d(x,x_n))_{n \geqslant 1}$. Since the coordinate map $f_n(x) = \frac{1}{D} d(x,x_n)$ is $\frac{1}{D}$-Lipschitz by the reverse triangle inequality, it is a continuous map from K into $[0,1]$. It follows that f is continuous from K into $[0,1]^{\mathbb{N}}$ equipped with the product topology,

Observe now that f is injective. Indeed, if $x \neq y$ then we can find $n \geqslant 1$ (by density) such that $d(y,x_n) \leqslant \frac{1}{4} d(x,y)$. Then, $f_n(x) - f_n(y) = \frac{1}{D}(d(x,x_n) - d(y,x_n)) \geqslant \frac{1}{D}(d(x,y) - 2d(y,x_n)) \geqslant \frac{1}{2D} d(x,y) > 0$. Therefore $f_n(x) \neq f_n(y)$ and in turn $f(x) \neq f(y)$, which proves injectivity.

Thus we have proved that f is a continuous bijection from the compact set K onto a compact set $f(K)$, and hence a homeomorphism between K and $f(K)$. It remains to show that $f(K)$ is closed, but this is true since $f(K)$ is compact and the product topology on $[0,1]^{\mathbb{N}}$ is Hausdorff since all the summands are Hausdorff (do you see why?)

□

Problem 4. *Let* $k \colon [0,1] \times [0,1] \to \mathbb{R}$ *be continuous. For* $f \in C([0,1])$ *define* $T(f) \colon [0,1] \to \mathbb{R}$ *by:*

$$T(f)(x) = \int_0^1 k(x,y) f(y) dy, \ x \in [0,1].$$

(1) Show that $T(C[0,1]) \subset C([0,1])$.

(2) A bounded set in $C([0,1])$ *is a set* S *for which there exists an* $R > 0$ *so that* $d_u(0,f) \leqslant R$ *for all* $f \in S$. *Show that* T *maps bounded sets into compact set.*

Solution. (1) This statement is a simple consequence of the automatic uniform continuity of continuous functions on compact metric spaces. Let $f \in C([0,1])$, $\varepsilon > 0$ and $x \in [0,1]$. Let $\|f\|_u = \sup_{x\in[0,1]} |f(x)|$. The product topology on $[0,1]^2$ is induced, for instance, by the metric $d_2((u,v),(\tilde{u},\tilde{v})) = \sqrt{(u-\tilde{u})^2 + (v-\tilde{v})^2}$. Since $k(\cdot,\cdot)$ is continuous on a compact metric space it is uniformly continuous. We can therefore find a $\delta > 0$ so that for all $(u,v),(\tilde{u},\tilde{v}) \in [0,1]^2$ with $\sqrt{(u-\tilde{u})^2 + (v-\tilde{v})^2} < \delta$ it follows that $|k(u,v) - k(\tilde{u},\tilde{v})| < \varepsilon/(1 + \|f\|_u)$. Therefore it follows for $z \in [0,1]$, with $|x - z| < \delta$ that

$$\begin{aligned} |T(f)(x) - T(f)(z)| &= \Big| \int_0^1 [k(x,y) - k(z,y)] f(y) dy \Big| \\ &\leqslant \|f\|_u \int_0^1 |k(x,y) - k(z,y)| dy \leqslant \|f\|_u \frac{\varepsilon}{1 + \|f\|_u} < \varepsilon. \end{aligned}$$

(2) Assume that $B \subset C(X)$ is bounded. Note that in the proof of (1) the δ only depended on ε and $\|f\|_u$. This implies that $\{T(f) \colon f \in B\}$ is equicontinuous.

Secondly if $f \in C([0,1])$,

$$\|T(f)\|_u \leqslant \sup_{x,y\in[0,1]} |k(x,y)| \cdot |f(x)| \leqslant \|f\|_u \sup_{x,y\in[0,1]} |k(x,y)|.$$

This implies that if $B \subset C(X)$ is bounded, then $T(B)$ is bounded.

From the theorem of Arzelà and Ascoli it follows that $\{T(f) : f \in B\}$ is totally bounded and thus $\overline{\{T(f) : f \in B\}}$ is compact.

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