REAL ANALYSIS MATH 608
HOMEWORK #5

Problem 1. *Consider the Banach space $C[0,1]$ consisting of all continuous, real valued functions on $[0,1]$, endowed with the uniform topology. For $f \in C[0,1]$, let*

$$\|f\|_L = |f(0)| + \sup_{0 \leqslant x < y \leqslant 1} \frac{|f(y) - f(x)|}{y - x}$$

Is the set $\{f \in C[0,1] \mid \|f\|_L < \infty\}$ dense in $C[0,1]$ for the uniform topology, or not? Justify your answer.

Problem 2. *Recall that a point is isolated in a topological space if $\{x\}$ is open, and a G_δ-set is a countable intersection of open sets. Let X be a non-empty Baire space that is T_1, and $Y \subset X$ that is countable, dense, and such that no point in Y is isolated in X. Show that*

(1) $X \setminus Y$ is a dense G_δ-set,
(2) Y is not a G_δ-set.
(3) Can $\mathbb{Q}$ be a G_δ-set in $\mathbb{R}$?

Problem 3 (Osgood's theorem)**.** *Let X be a complete metric space and Y be a metric space. Let $\mathcal{F} \subset C(X,Y)$ such that the set $\{f(x)\colon\ f \in \mathcal{F}\} \subset Y$ is bounded for each $x \in X$. Then, there is a non-empty open set $U \subset X$ such that $\{f(x)\colon\ f \in \mathcal{F}, x \in U\} \subset Y$ is bounded.*

Problem 4. *A topological space (X,τ) is* locally compact *if every $x \in X$ admits a neighborhood basis consisting of compact sets.*

(1) Assuming that (X,τ) is Hausdorff, show that X is locally compact if and only if every point admits a compact neighborhood.
(2) Show that a Hausdorff compact space is locally compact.
(3) Show that every Hausdorff locally compact space is a Baire space.

Problem 5 (Completeness of $L(X,Y)$)**.** *Let X and Y be normed linear spaces and $L(X,Y)$ the vector space of bounded linear maps from X into Y.*

(1) Show that $\|T\| = \sup_{x \in X, \|x\|_X \leqslant 1} \|T(x)\|_Y$ defines a norm on $L(X,Y)$.
(2) Show that

$$\|T\| = \sup\Big\{\frac{\|T(x)\|_Y}{\|x\|_X} : x \neq 0\Big\} = \inf\{C \geqslant 0 : \forall x \in X \quad \|T(x)\|_Y \leqslant C\|x\|_X\}.$$

(3) Show that if Y is a Banach space then $L(X,Y)$ is complete, and thus also a Banach space.