REAL ANALYSIS MATH 608
HOMEWORK #5

Problem 1. *Consider the Banach space* $C[0,1]$ *consisting of all continuous, real valued functions on* $[0,1]$*, endowed with the uniform topology. For* $f \in C[0,1]$*, let*

$$\|f\|_L = |f(0)| + \sup_{0\leqslant x<y\leqslant 1} \frac{|f(y)-f(x)|}{y-x}$$

Is the set $S = \{f \in C[0,1] \mid \|f\|_L < \infty\}$ *dense in* $C[0,1]$ *for the uniform topology, or not? Justify your answer.*

Solution. This is an archi-typical application of Stone-Weierstrass theorem.

S contains the constants.: This is clear.

S separates points: You could argue that S contains the polynomials (since their derivatives are polynomials, thus continuous functions on a compact space, and hence their derivative are bounded, which makes them Lipschitz maps) and polynomials separate points. Or more simply, you could construct explicitely continuous piecewise linear functions separating two distinct points $x < y$ (e.g. a function that is 1 on $[0, x]$ and 0 on $[y, 1]$ and extended continuously and linearly between x and y. The Lipschitz constant of this map is clearly $\frac{1}{d(x,y)} < \infty$.

S is an algebra: Since continuity is preserved under the algebra operations it is sufficient to show that

- $\|\lambda f\|_L = |\lambda|\|f\|_L$; but this is immediate.
- $\|f + g\|_L \leqslant \|f\|_L + \|g\|_L$; this follows from the triangle inequality.
- $\max \|f\|_L, \|g\|_L < \infty \implies \|fg\|_L < \infty$; this needs a bit more work but follows from the following decomposition

$$\frac{|(fg)(y)-(fg)(x)|}{y-x} = \frac{|(f(y)-f(x))g(y)+f(x)(g(y)-g(x))|}{y-x} \leqslant \frac{|(f(y)-f(x))|}{y-x}|g(y)| + |f(x)|\frac{|g(y)-g(x)|}{y-x},$$

which gives $\|fg\|_L \leqslant |f(0)||g(0)| + \|g\|_u \sup_{0\leqslant x<y\leqslant 1} \frac{|f(y)-f(x)|}{y-x} + \|f\|_u \sup_{0\leqslant x<y\leqslant 1} \frac{|g(y)-g(x)|}{y-x}$, which is clearly finite if $\max \|f\|_L, \|g\|_L < \infty$ (and since f, g are continuous on a compact space and hence bounded)

By Stone-Weierstrass theorem S is dense in $C[0,1]$. □

Problem 2. *Recall that a point is isolated in a topological space if* $\{x\}$ *is open, and a* G_δ*-set is a countable intersection of open sets. Let* X *be a non-empty Baire space that is* T_1*, and* $Y \subset X$ *that is countable, dense, and such that no point in* Y *is isolated in* X*. Show that*

(1) $X \setminus Y$ *is a dense* G_δ*-set,*
(2) Y *is not a* G_δ*-set.*
(3) Can $\mathbb{Q}$ *be a* G_δ*-set in* $\mathbb{R}$*?*

Solution. (1) $Y = \cup_{y\in Y}\{y\}$ and this union is countable. Since X is T_1 the singletons are closed. Since no point in Y is isolated $\{y\}$ has empty interior for any $y \in Y$. Therefore, Y is a countable intersections of closed sets with empty interior, and by taking the complement $X \setminus Y$ is a countable intersection of dense open sets, and hence a dense G_δ-set since X is a Baire space.

(2) Assume that $Y = \cap_n U_n$ with U_n open. Given $n \geqslant 1$, $Y \subset U_n$, and U_n is dense because Y is dense. So Y is a countable intersection of dense open sets, and $X \setminus Y$ is the countable union of closed sets with

empty interior. In (1) we showed that Y is also a countable intersections of closed sets with empty interior, so $X = X \setminus Y \cup Y$ is a countable intersections of closed sets with empty interior, nad hence has empty interior since X is Baire, a contradiction.

(3) No by (2).

□

Problem 3 (Osgood's theorem). *Let X be a complete metric space and Y be a metric space. Let $\mathcal{F} \subset C(X,Y)$ such that the set $\{f(x) \colon f \in \mathcal{F}\} \subset Y$ is bounded for each $x \in X$. Then, there is a non-empty open set $U \subset X$ such that $\{f(x) \colon f \in \mathcal{F}, x \in U\} \subset Y$ is bounded.*

Solution. This is a prototypical application of Baire Category Theorem. Fix $y_0 \in Y$, and for $n \geqslant 1$, let

$$F_n = \{x \in X \colon \forall f \in \mathcal{F},\ d_Y(y_0, f(x)) \leqslant n\}.$$

Then, by continuity of the distance function and the functions in $\mathcal{F}$, F_n is closed and we infer from the assumption that $X = \cup_{n=1}^{\infty} F_n$. Since X is complete, BCT tells us that there must be a $n_0 \in \mathbb{N}$ such that $F_{n_0}^{\circ} \neq \emptyset$. But if $f \in \mathcal{F}$ and x is in the open set $F_{n_0}^{\circ}$, we have $d_Y(y_0, f(x)) \leqslant n_0$, which is exactly saying that $\{f(x) \colon f \in \mathcal{F}, x \in F_{n_0}^{\circ}\} \subset Y$ is bounded. □

Problem 4. *A topological space (X, τ) is* locally compact *if every $x \in X$ admits a neighborhood basis consisting of compact sets.*

(1) Assuming that (X, τ) is Hausdorff, show that X is locally compact if and only if every point admits a compact neighborhood.

(2) Show that a Hausdorff compact space is locally compact.

(3) Show that every Hausdorff locally compact space is a Baire space.

Solution.

(1) If X is locally compact then by definition every point admits a compact neighborhood (and we do not need Hausdorfness).

Assume now that (X, τ) is Hausdorff, and that every point admits a compact neighborhood. Let U be an open neighborhood of $x \in X$ and let K be a compact neighborhood of x. The problem here is that K might not be included in U. We will use that fact that Hausdorff compact spaces are normal to find a compact neighborhood that is included in U. To do so we first observe that $K^{\circ} \cap U$ is an open neighborhood of x whose closure is compact. Indeed, $\overline{K^{\circ} \cap U} \subset \overline{K}$, but K is closed since compact in a Hausdorff space, and hence $\overline{K^{\circ} \cap U}$ is compact as a closed subset of a compact space. Let $V = K^{\circ} \cap U$, we will work in the closed subset $\overline{V}$ equipped with its subspace topology. Since $x \in V$, it follows that $x \notin V^c \cap \overline{V}$. Since $\overline{V}$ is compact Hausdorff for the subspace topology, and $V^c \cap \overline{V}$ is closed (for the subspace topology on $\overline{V}$ and in X), and $x \in \overline{V}$, by normality we can find disjoint relatively open sets W_1, W_2 such $V^c \cap \overline{V} \subset W_1$ and $x \in W_2$. The tricky point is to observe that W_2 is open in X since it is in fact a subset of the open set V. If you draw a diagram it is clear, but rigorously it follows from the fact that by disjointness of W_1 and W_2 (which are subsets of $\overline{V}$),

$$W_2 \subset \overline{V} \setminus W_1 = \overline{V} \cap (V^c \cap \overline{V})^c = \overline{V} \cap (V \cup (\overline{V}^c) = V.$$

Since $\overline{V} \setminus W_1$ is relatively closed and $W_2 \subset \overline{V} \setminus W_1$, the relative closure of W_2 is a subset of $\overline{V} \setminus W_1 \subset V = K^{\circ} \cap U \subset U$. But the relative closure of W_2 coincides with the closure of W_2 in X since $\overline{V}$ is closed. Therefore, $\overline{W_2}$ is a compact neigborhood of x that is included in U.

(2) This is an immediate consequence of (1).

(3) Let $(U_n)_n$ be a sequence of open and dense sets. Let's try to mimic the proof of BCT in the metric setting. Let W be a non-empty open set. Then $W \cap U_1$ is non empty by density and let $x_1 \in W \cap U_1$. Since $W \cap U_1$ is open and non empty, by local compactness there is a compact neighborhood K_1 such that $\emptyset \neq K_1^\circ \subset K_1 \subset W \cap U_1$. Since K_1° is open and non-empty, one can find a compact neighborhood K_2 such that $\emptyset \neq K_2^\circ \subset K_2 \subset K_1^\circ \cap U_2 \subset W \cap U_2$ (again by density and local compactness). Continuing recursively, we can find a decreasing sequence of compact neighborhood $(K_n)_n$ such that for all $n \geqslant 1$

$$K_n \subset W \cap U_n.$$

Since $(K_n)_n$ is a decreasing sequence of compact spaces the intersection $\cap_{n=1}^\infty K_n$ is non-empty (you might want to prove this and here we need Hausdorfness). Since $\emptyset \neq \cap_n K_n \subset W \cap \cap_{n=1}^\infty U_n$, it follows that $\cap_n U_n$ is dense.

□

Problem 5 (Completeness of $L(X,Y)$). *Let X and Y be normed linear spaces and $L(X,Y)$ the vector space of bounded linear maps from X into Y.*

(1) Show that $\|T\| = \sup_{x \in X, \|x\|_X \leqslant 1} \|T(x)\|_Y$ defines a norm on $L(X,Y)$.

(2) Show that

$$\|T\| = \sup\Big\{\frac{\|T(x)\|_Y}{\|x\|_X} : x \neq 0\Big\} = \inf\{C \geqslant 0 : \forall x \in X \quad \|T(x)\|_Y \leqslant C\|x\|_X\}.$$

(3) Show that if Y is a Banach space then $L(X,Y)$ is complete, and thus also a Banach space.

Solution. (1) $\|\cdot\|$ is a norm.

Definitness:

$$\|T\| = 0 \iff \sup_{x \in X, \|x\| \leqslant 1} \|T(x)\| = 0 \iff \forall x \in X, \|x\| \leqslant 1, \quad T(x) = 0$$

But this is equivalent to $\forall x \in X, T(x) = 0$ (by scalar homogeneity of T) and this means that $T = 0$.

homogeneity: $\|\lambda T\| = \sup_{x \in X, \|x\| \leqslant 1} \|\lambda T(x)\| = |\lambda| \sup_{x \in X, \|x\| \leqslant 1} \|T(x)\| = |\lambda| \|T\|$.

triangle inequality:

$$\begin{aligned} \|T + S\| &= \sup_{x \in X, \|x\| \leqslant 1} \|T(x) + S(x)\| \\ &\leqslant \sup_{x \in X, \|x\| \leqslant 1} \|T(x)\| + \|S(x)\| \\ &\leqslant \sup_{x \in X, \|x\| \leqslant 1} \|T(x)\| + \sup_{x \in X, \|x\| \leqslant 1} \|S(x)\| = \|T\| + \|S\|. \end{aligned}$$

(2) We will show that

$$\sup\Big\{\frac{\|T(x)\|}{\|x\|} : x \neq 0\Big\} \overset{(1)}{\leqslant} \|T\| \overset{(2)}{\leqslant} \inf\{C \geqslant 0 \colon \|T(x)\| \leqslant C\|x\|, \forall x \in X\} \overset{(3)}{\leqslant} \sup\Big\{\frac{\|T(x)\|}{\|x\|} : x \neq 0\Big\}.$$

(1) $\sup\Big\{\frac{\|T(x)\|}{\|x\|} \colon x \neq 0\Big\} \leqslant \|T\|$**:**

Follows from the observation that $\Big\{\frac{\|T(x)\|}{\|x\|} \colon x \neq 0\Big\} = \Big\{\|T(x)\| \colon \|x\| = 1\Big\}$ since for $x \in X \setminus \{0\}$, $\|T(x)\|/\|x\| = \|T(x/\|x\|)\|$.

(2) $\|T\| \leqslant \inf\{C \geqslant 0 \colon \|T(x)\| \leqslant C\|x\|, \forall x \in X\}$**:**

Assume that $C \geqslant 0$ is such that $\forall x \in X \quad \|T(x)\| \leqslant C\|x\|$. Then $\|T(x)\| \leqslant C$ whenever $\|x\| \leqslant 1$. The inequality follows by taking the sup over $x \in B_X$ first, and then the inf over C.

(3) $\inf\{C \geqslant 0 \colon \|T(x)\| \leqslant C\|x\|, \forall x \in X\} \leqslant \sup\left\{\frac{\|T(x)\|}{\|x\|} \colon x \neq 0\right\}$**:**

This is clear, since by definition $\|T(x)\| \leqslant \sup\left\{\frac{\|T(x)\|}{\|x\|} \colon x \neq 0\right\} \cdot \|x\|$, for all $x \in X$.

(3) Assume that Y is complete, and let $\sum_n T_n$ be absolutely convergent in $L(X,Y)$, *i.e.,*

$$C = \sum_{n=1}^{\infty} \|T_n\| = \sum_{n=1}^{\infty} \sup_{x \in B_X} \|T_n(x)\| < \infty.$$

This implies that for fixed $x \in X$ the series $\sum_{n=1}^{\infty} T_n(x)$ is absolutely convergent and thus (since Y is complete) convergent to an element which we denote by $S(x)$. It is then easy to verify that the map $x \mapsto S(x)$ is a linear operator and that for $x \neq 0$

$$\|S(x)\| = \|x\| \cdot \|S(x/\|x\|)\| = \|x\| \cdot \left\| \sum_{n=1}^{\infty} T_n(x/\|x\|) \right\| \leqslant \|x\| \sum_{n=1}^{\infty} \|T_n\| = C\|x\|,$$

which implies that $S \in L(X,Y)$. Moreover, for $x \in B_X$, and $n \in \mathbb{N}$

$$\left\| S(x) - \sum_{j=1}^{n} T_j(x) \right\| \leqslant \sum_{j=n+1}^{\infty} \|T_j(x)\| \leqslant \sum_{j=n+1}^{\infty} \|T_j\|.$$

and thus

$$\left\| S - \sum_{j=1}^{n} T_j \right\| \leqslant \sum_{j=n+1}^{\infty} \|T_j\| \to_{n \to \infty} 0.$$

Therefore $\sum_n T_n$ converge to S in $L(X,Y)$, and we conclude that $L(X,Y)$ is complete for the operator norm.

□