REAL ANALYSIS MATH 608
HOMEWORK #6

Problem 1. *Let X be a finite dimensional vector space (over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$) and let $e_1, e_2, \dots, e_n$ be an algebraic basis of X. For $x = \sum_{j=1}^n a_j e_j \in X$, consider the norm $\|x\|_1 \overset{\text{def}}{=} \sum_{j=1}^n |a_j|$.*

(1) Show that the unit sphere of $(X, \|\cdot\|_1)$, i.e. the set $S_X \overset{\text{def}}{=} \{x \in X \colon \|x\|_1 = 1\}$ is compact in the topology defined by $\|\cdot\|_1$.
(2) Show that all norms on X are equivalent, i.e. for every norm $\|\cdot\|$ on X there are constant $C_X, c_X > 0$, such that $c_X\|x\| \leqslant \|x\|_1 \leqslant C_X\|x\|$, for all $x \in X$.

Problem 2.

(1) Assume that $(X, \|\cdot\|)$ is a normed vector space and that Y is a closed proper subspace of X. Show that for any $\varepsilon > 0$ there is an $x \in X$ such that $\|x\| = 1$ and $d(x, Y) > 1 - \varepsilon$.
(2) Let $(X, \|\cdot\|)$ be an infinite-dimensional normed vector space and $\varepsilon \in (0,1)$. Show that there is a sequence $\{x_n\}_{n=1}^\infty \subset X$, with $\|x_n\| = 1$, for all $n \in \mathbb{N}$, and $\|x_j - x_i\| \geqslant (1 - \varepsilon)$, whenever $i \neq j$.
(3) What can you deduce from (2) about the topological properties of the unit ball $B_X = \{x \in X, \|x\| \leqslant 1\}$ of any infinite-dimensional normed vector space X.

Hint: For (2) you can use the fact that every finite-dimensional subspace is closed (which essentially follows from Problem 1, do you see why?).

Problem 3. *Let X and Y be two Banach spaces. The adjoint of a map $T \colon X \to Y$ is the map $T^* \colon Y^* \to X^*$ defined by $T^*(y^*)(x) = y^*(Tx)$, for all $y^* \in Y^*, x \in X$.*

(1) Show that if $T \colon X \to Y$ is linear and bounded, then the adjoint $T^* \colon Y^* \to X^*$ is also linear and bounded and $\|T\| = \|T^*\|$.
(2) Show that if T is an isomorphism (resp. onto isometry) then T^* is an isomorphism (resp. onto isometry).

Problem 4. *Show that a linear functional f on a normed vector space X is bounded if and only if* $\ker(f) \overset{\text{def}}{=} f^{-1}(\{0\})$ *is closed.*

Hint: For the non-trivial implication you might want to use a Hahn-Banach argument.

Problem 5. *Let $\{x_n\}_{n=1}^\infty$ be a sequence in a Banach space X that converges to $x \in X$ for the weak topology. Show that*

(1) $\{x_n\}_{n=1}^\infty$ is bounded,
(2) $\|x\| \leqslant \liminf_{n\to\infty} \|x_n\|$,

Let $\{x_n^*\}_{n=1}^\infty$ be a sequence in the dual X^* of a Banach space X that converges to $x^* \in X^*$ for the weak-$*$ topology. Show that

(1) $\{x_n^*\}_{n=1}^\infty$ is bounded,
(2) $\|x^*\| \leqslant \liminf_{n\to\infty} \|x_n^*\|$.

Bonus question: Can you spot where and/or if completeness of X is needed at all?

Hint:
3 out of the 4 statements are consequences of a consequence of Baire category theorem and/or of a consequence of Hahn-Banach theorem.