REAL ANALYSIS MATH 608
HOMEWORK #6

Problem 1. *Let X be a finite dimensional vector space (over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$) and let $e_1, e_2, \ldots, e_n$ be an algebraic basis of X. For $x = \sum_{j=1}^n a_j e_j \in X$, consider the norm $\|x\|_1 \stackrel{\text{def}}{=} \sum_{j=1}^n |a_j|$.*

(1) Show that the unit sphere of $(X, \|\cdot\|_1)$, i.e. the set $S_X \stackrel{\text{def}}{=} \{x \in X \colon \|x\|_1 = 1\}$ is compact in the topology defined by $\|\cdot\|_1$.

(2) Show that all norms on X are equivalent, i.e. for every norm $\|\cdot\|$ on X there are constant $C_X, c_X > 0$, such that $c_X\|x\| \leqslant \|x\|_1 \leqslant C_X\|x\|$, for all $x \in X$.

Solution. (1) Let $x^{(k)} = \sum_{j=1}^n a_j^{(k)} e_j \in S_X$, for $k \in \mathbb{N}$ note that for each $j \in \{1, 2, \ldots n\}$, $(a_j^{(k)})_{k\in\mathbb{N}} \subset \{\xi \in \mathbb{K} : |\xi| = 1\}$. We find therefore an infinite $N \subset \mathbb{N}$ so that

$$a_j = \lim_{k\in\mathbb{N}, k\to\infty} a_j^{(k))}$$

exists for all $j \in \{1, 2 \ldots\}$. Let $x = \sum_{j=1}^n a_j e_j$. Then

$$\|x\|_1 = \sum_{j=1}^n |a_j| = \lim_{k\in\mathbb{N}, k\to\infty} \sum_{j=1}^n |a_j^{(k)}| = 1,$$

and

$$\|x - x^{(k)}\|_1 = \sum_{j=1}^n |a_j - a_j^{(k)}| = 1 \to_{k\in\mathbb{N}, k\to\infty} 0.$$

Thus, every sequence in S_X has a subsequence which converges to an element of S_X. Thus S_X is compact.

(2) Let $\|\cdot\|$ be any norm on X. Put $C = \max_{j=1,2\ldots n} \|e_j\|$. Then (property of norm) $0 < C < \infty$. Let T be the identity on X, but think of it as a linear map from $(X, \|\cdot\|_1)$ to $(X, \|\cdot\|)$.

Since for $x = \sum_{j=1}^n a_j e_j \in X$

$$\|x\| = \|T(x)\| = \Big\| \sum_{j=1}^n a_j e_j \Big\| \leqslant \sum_{j=1}^n |a_j| \|e_j\| \leqslant C\|x\|_1$$

T is a bounded linear operator with $\|T\| \leqslant C$. This implies that the image of $S_X = \{x \in X : \|x\|_1\}$ is compact in $(X, \|\cdot\|)$. And since $0 \notin S_X$ and since $\|\cdot\|$ is a $\|\cdot\|$-continuous function on X (simply meaning that if x_n converges to x in $(X, \|\cdot\|)$ then $\|x_n\|$ converges to $\|x\|$) it follows that $c := \min\{\|x\| : x \in S_X\}$ exists and $c > 0$. We deduce that for $x \in X$

$$C\|x\|_1 \geqslant \|x\| = \|x\|_1 \Big\| \frac{x}{\|x\|_1} \Big\| \geqslant c\|x\|_1,$$

which proves our claim (c).

□

Problem 2.

(1) Assume that $(X, \|\cdot\|)$ is a normed vector space and that Y is a closed proper subspace of X. Show that for any $\varepsilon > 0$ there is an $x \in X$ such that $\|x\| = 1$ and $d(x, Y) > 1 - \varepsilon$.

(2) Let $(X, \|\cdot\|)$ be an infinite-dimensional normed vector space and $\varepsilon \in (0, 1)$. Show that there is a sequence $\{x_n\}_{n=1}^\infty \subset X$, with $\|x_n\| = 1$, for all $n \in \mathbb{N}$, and $\|x_j - x_i\| \geqslant (1 - \varepsilon)$, whenever $i \neq j$.

(3) *What can you deduce from (2) about the topological properties of the unit ball $B_X = \{x \in X, \|x\| \leqslant 1\}$ of any infinite-dimensional normed vector space X.*

Hint: For (2) you can use the fact that every finite-dimensional subspace is closed (which essentially follows from Problem 1, do you see why?).

Solution. (1) Y is a proper and closed subspace, thus not dense in X. So choose $x_0 \in X$, with $\mathrm{dist}(x_0, Y) = \inf_{y\in Y} \|x_0 - y\| > 0$. Given $\varepsilon > 0$ choose $y_0 \in Y$ so that

$$\mathrm{dist}(x_0, Y) = \inf_{y\in Y} \|x_0 + y\| > \|x_0 + y_0\|(1-\varepsilon).$$

Finally let $x = (x_0 + y_0)/\|x_0 + y_0\|$. It follows that $\|x\| = 1$ and

$$\begin{aligned}\mathrm{dist}(x, Y) &= \|x_0 + y_0\|^{-1} \cdot \mathrm{dist}(x_0 + y_0, Y)\\ &= \|x_0 + y_0\|^{-1} \cdot \inf_{y\in Y} \|x_0 + y_0 + y\|\\ &= \|x_0 + y_0\|^{-1} \cdot \inf_{y\in Y} \|x_0 + y\| > 1 - \varepsilon.\end{aligned}$$

(2) By induction we choose $x_1, x_2, \dots$ in S_X so that $\|x_j - x_i\| \geqslant (1-\varepsilon)$ whenever $i \neq j$. Choose x_1 arbitrary and assume that $x_1, \dots x_n$ have been chosen. Let Y be the space generated by $x_1, \dots, x_n$. Y is a closed and proper subspace of X. By (1), there is an x_{n+1} so that $\|x_{n+1}\| = 1$ and $d(x_{n+1}, Y) > (1-\varepsilon)$. Thus for all $i = 1, 2 \dots n$,

$$\|x_{n+1} - x_i\| \geqslant \mathrm{dist}(x_{n+1}, Y) > (1-\varepsilon).$$

(3) It is not compact since in a compact metric space there is no infinite separated sequence.

□

Problem 3. *Let X and Y be two Banach spaces. The adjoint of a map $T : X \to Y$ is the map $T^* : Y^* \to X^*$ defined by $T^*(y^*)(x) = y^*(Tx)$, for all $y^* \in Y^*, x \in X$.*

(1) *Show that if $T : X \to Y$ is linear and bounded, then the adjoint $T^* : Y^* \to X^*$ is also linear and bounded and $\|T\| = \|T^*\|$.*

(2) *Show that if T is an isomorphism (resp. onto isometry) then T^* is an isomorphism (resp. onto isometry).*

Proof. First note that for $y_1^*, y_2^* \in Y^*$, $\lambda \in \mathbb{K}$ and $x \in X$ we have

$$T^*(y_1^* + y_2^*)(x) = (y_1^* + y_2^*)(T(x)) = y_1^*(T(x)) + y_2^*(T(x)) = (T^*(y_1^*) + T^*(y_2^*))(x)$$

and

$$T^*(\lambda y_1^*)(x) = (\lambda y_1^*)(T(x)) = \lambda T^*(y_1^*))(x).$$

Thus T^* is linear.

Secondly we show $\|T^*\| = \|T\|$.

$$\|T^*\| = \sup_{y^*\in B_{Y^*}} \|T^*(y^*)\| = \sup_{y^*\in B_{Y^*}} \sup_{x\in B_X} \left|y^*(T(x))\right| = \sup_{x\in B_X} \left\|T(x)\right\| = \|T\|$$

(In the third "=" we are using the Hahn Banach Theorem).

Now assume that T is an onto isomorphism and thus T is surjective for some numbers $0 < c \leqslant C$ we have

$$c\|x\| \leqslant \|T(x)\| \leqslant C\|x\| \text{ for all } x \in X$$

(with $c = C = 1$ if T is an isometry).

Let $y^* \in Y^*$. Since $cB_Y \subset T(B_X) \subset CB_Y$, it follows that

$$\|T^*(y^*)\| = \sup_{x\in B_X} \|y^*(T(x))\| = \sup_{y\in T(B_X)} \|y^*(y)\|$$

$$\begin{cases} \leqslant \sup_{y\in Y, \|y\|\leqslant C} |y^*(y)| = C \sup_{y\in Y, \|y\|\leqslant 1} \left|y^*(y)\right| = C\|y^*\|. \\ \geqslant \sup_{y\in Y, \|y\|\leqslant c} |y^*(y)| = c \sup_{y\in Y, \|y\|\leqslant 1} \left|y^*(y)\right| = c\|y^*\|. \end{cases}$$

Also the proof shows that if T is an isometry, and thus we can choose $c = C = 1$, T^* is also an isometry.

We still have to show that T is surjective. Let $x^* \in X^*$ and put $y^* = x^* \circ T^{-1}$. Then

$$T^*(y^*) = y^* \circ T = x^* \circ T^{-1} \circ T = x^*.$$

□

Problem 4. *Show that a linear functional f on a normed vector space X is bounded if and only if* $\ker(f) \overset{\text{def}}{=} f^{-1}(\{0\})$ *is closed.*

Hint: For the non-trivial implication you might want to use a Hahn-Banach argument.

Solution. $\Rightarrow$ is clear since bounded linear functionals on a normed linear space are continuous, and the preimages of closed sets under continuous maps are closed.

$\Leftarrow$ W.l.o.g. f is not the zero functional (which is clearly bounded). So let $x_0 \in X$ with $f(x_0) \neq 0$, say $f(x_0) = 1$ (after multiplication with the right scalar). Then we can write every $x \in X$ as $x = f(x)x_0 + y$ with $y \in Y = f^{-1}(\{0\})$ (simply note that $x - f(x)x_0$ must be in $f^{-1}(\{0\})$). Now use Theorem of Hahn-Banach (more precisely Theorem 7.2.6 from the notes) to get a linear bounded functional g on X so that $g|_Y = 0$ and $g(x_0) \neq 0$. After multiplying g with the right scalar we may assume $g(x_0) = 1$. Now we claim that $g = f$, and thus f must be continuous.

Indeed for all $x \in X$ we find $y \in Y$ with $x = f(x)x_0 + y$, and thus

$$g(x) = g(f(x)x_0 + y) = f(x)g(x_0) + g(y) = f(x).$$

□

Problem 5. *Let $\{x_n\}_{n=1}^\infty$ be a sequence in a Banach space X that converges to $x \in X$ for the weak topology. Show that*

(1) $\{x_n\}_{n=1}^\infty$ is bounded,

(2) $\|x\| \leqslant \liminf_{n\to\infty} \|x_n\|$,

Let $\{x_n^*\}_{n=1}^\infty$ be a sequence in the dual X^* of a Banach space X that converges to $x^* \in X^*$ for the weak-$*$ topology. Show that

(1) $\{x_n^*\}_{n=1}^\infty$ is bounded,

(2) $\|x^*\| \leqslant \liminf_{n\to\infty} \|x_n^*\|$.

Bonus question: Can you spot where and/or if completeness of X is needed at all?

Hint:
3 out of the 4 statements are consequences of a consequence of Baire category theorem and/or of a consequence of Hahn-Banach theorem.

Solution. These statements are classical and extremely useful consequences of the UBP and of HBT.

(1) For $n \geqslant 1$, consider the map $T_n \colon X^* \to \mathbb{F}$ given by $T_n(x^*) = x^*(x_n)$. Clearly, T_n is linear and $|T_n(x^*)| = |x^*(x_n)| \leqslant \|x^*\| \cdot \|x_n\|$ is bounded with $\|T_n\| \leqslant \|x_n\|$. In fact $\|T_n\| = \|x_n\|$, since it follows from HBT that there is $x_n^* \in X^*$, with $\|x_n^*\| = 1$ and $x_n^*(x_n) = \|x_n\|$. Now, for all $x^* \in X^*$, $(x^*(x_n))_n$ converges by assumption, and hence $(T_n(x^*))_n$ is bounded for all $x^* \in X^*$. By the UBP (here we need the completeness of X^* which is automatic), we have that $\sup_n \|T_n\| = \sup_n \|x_n\| < \infty$.

(2) By HBT let $x^* \in S_{X^*}$ such that $x^*(x) = \|x\|$. Then, $x^*(x_n) \leqslant \|x^*\| \cdot \|x_n\| = \|x_n\|$ and taking $\liminf$ on both sides we have $\|x\| = x^*(x) \leqslant \liminf_n \|x_n\|$ (since $\{x_n\}_{n=1}^\infty$ converges weakly to x).

(1) For $n \geqslant 1$, the maps $x_n^* \colon X \to \mathbb{F}$ are linear and bounded by assumption. Now, for all $x \in X$, $(x_n^*(x))_n$ converges by assumption, and hence $(x_n^*(x))_n$ is bounded for all $x \in X$. By the UBP (here we use that X is a Banach space), we have that $\sup_n \|x_n^*\| < \infty$.

(2) Let $x \in S_X$. Then, $|x_n^*(x)| \leqslant \|x_n^*\| \cdot \|x\| = \|x_n^*\|$. Since $\{x_n^*\}_{n=1}^\infty$ converges weak-$*$ to x^*, taking $\liminf$ on both sides we have $|x^*(x)| \leqslant \liminf_n \|x_n\|$ (by continuity of the module), and hence $\|x^*\| \leqslant \liminf_n \|x_n\|$.

□