REAL ANALYSIS MATH 608
HOMEWORK #7

Problem 1. *Let X be a normed vector space and $Y \subset X$ be a vector subspace such that $\overline{Y} \neq X$ (the closure is with respect tot the norm topology). Show that there exists $x^* \in X^*$, $x^* \neq 0$, and $x^*(y) = 0$ for all $y \in Y$.*

Hint: Use a geometric form of Hahn-Banach theorem.

Problem 2. *Let X be a normed space and $C \subset X$ be a subset.*

(1) Show that if C is weakly-closed then it is norm-closed.
(2) Show that if C is convex and norm-closed then it is weakly-closed.
(3) (Mazur's Theorem) Show that if C is convex, then $\overline{C}^{\|\cdot\|} = \overline{C}^{w}$, i.e. weak and norm closure coincide for convex subsets.
*(4) Show that if $\{x_n\}_{n=1}^{\infty}$ converges **weakly** to $x \in X$, then there is a sequence of convex combination of elements in $\{x_n : n \in \mathbb{N}\}$ that converges in **norm** to x.*

Hint: For (2) use a geometric form of Hahn-Banach theorem. For (4) consider the convex hull of $\{x_n : n \in \mathbb{N}\}$, i.e. the smallest convex set that contains $\{x_n : n \in \mathbb{N}\}$, and show that the convex hull of a set consists of all the convex combinations of elements in the set.

Problem 3. *Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and define*

$$c_0(\mathbb{N};\mathbb{F}) \stackrel{\text{def}}{=} \{\{x_n\}_{n=1}^{\infty} \subset \mathbb{F} : \lim_{n\to\infty} x_n = 0\}.$$

For $x = \{x_n\}_{n=1}^{\infty} \in c_0(\mathbb{N};\mathbb{F})$ define $\|x\|_\infty = \sup_{n\in\mathbb{N}} |x_n|$.

(1) Show that $(c_0(\mathbb{N};\mathbb{F}), \|\cdot\|_\infty)$ is a normed vector space.
(2) Show that $(c_0(\mathbb{N};\mathbb{F}), \|\cdot\|_\infty)$ is a Banach space.

Problem 4. *Let $(X, \mathcal{M}, \mu)$ be a measure space and denote $L_p(\mu) := L_p(X, \mathcal{M}, \mu)$ for $1 \leqslant p < \infty$.*

(1) If $f \in L_2(\mu)$, $g \in L_3(\mu)$ and $h \in L_6(\mu)$, show that $fgh \in L_1(\mu)$ and

$$\|fgh\|_1 \leqslant \|f\|_2 \cdot \|g\|_3 \cdot \|h\|_6.$$

(2) Formulate a generalization for the product of finitely many functions, and prove it.

Problem 5. *Let X be an infinite dimensional normed vector space. Show that $\overline{S_X}^{w} = B_X$, i.e. the weak-closure of the unit sphere of X is the unit ball of X.*

Hint: Use Mazur's Theorem for one inclusion. For the other inclusion, show that every weak-neighborhood of a point $x_0 \in B_X \setminus S_X$ contains an affine line which eventually intersects the sphere.