REAL ANALYSIS MATH 608
HOMEWORK #7

Problem 1. *Let X be a normed vector space and $Y \subset X$ be a vector subspace such that $\overline{Y} \neq X$ (the closure is with respect tot the norm topology). Show that there exists $x^* \in X^*$, $x^* \neq 0$, and $x^*(y) = 0$ for all $y \in Y$.*

Hint: Use a geometric form of Hahn-Banach theorem.

Solution. Let $x_0 \in X \setminus \overline{Y}$, and apply the second geometric form of the HBT to the closed convex set $C_1 = \overline{Y}$, and the compact convex set $C_2 = \{x_0\}$. Then, there is a non-zero functional $x^* \in X^*$, and $\alpha \in \mathbb{R}$ such that $x^*(x) < \alpha < x^*(x_0), \forall y \in \overline{Y}$. If there exists $y_0 \in Y$ such that $x^*(y_0) \neq 0$ we can assume w.l.o.g. that $x^*(y_0) > 0$ and then $x^*(ty_0) = tx^*(y_0) < \alpha$ for all $t > 0$, contradiction! Therefore, $x^*_{|Y} = 0$. □

Problem 2. *Let X be a normed space and $C \subset X$ be a subset.*

(1) Show that if C is weakly-closed then it is norm-closed.
(2) Show that if C is convex and norm-closed then it is weakly-closed.
(3) (Mazur's Theorem) Show that if C is convex, then $\overline{C}^{\|\cdot\|} = \overline{C}^{w}$, i.e. weak and norm closure coincide for convex subsets.
*(4) Show that if $\{x_n\}_{n=1}^{\infty}$ converges **weakly** to $x \in X$, then there is a sequence of convex combination of elements in $\{x_n\colon n \in \mathbb{N}\}$ that converges in **norm** to x.*

Hint: For (2) use a geometric form of Hahn-Banach theorem. For (4) consider the convex hull of $\{x_n\colon n \in \mathbb{N}\}$, i.e. the smallest convex set that contains $\{x_n\colon n \in \mathbb{N}\}$, and show that the convex hull of a set consists of all the convex combinations of elements in the set.

Solution. (1) By definition since every weakly open set is norm open (and then take complements).

(2) If $C = X$ there is nothing to show. So let $x_0 \in X \setminus C$. By the second form of the geometric HBT, there is a non-zero functional $x^* \in X^*$, and $\alpha \in \mathbb{R}$ such that $x^*(x_0) < \alpha < x^*(z), \forall z \in C$. Therefore, $U = (x^*)^{-1}((-\infty, \alpha))$ is a weak open set that contains x_0 and such that $U \subset X \setminus C$ since $U \cap C = \emptyset$. This means that $X \setminus C$ is weakly open and thus C is weakly closed.

(3) Every weakly closed set is norm closed, and hence $\overline{C}^{\|\cdot\|} \subset \overline{C}^{w}$. It is easy to see that $\overline{C}^{\|\cdot\|}$ is convex and obviously closed, and since $C \subset \overline{C}^{\|\cdot\|}$ which is weakly closed by (2) we have that $\overline{C}^{w} \subset \overline{C}^{\|\cdot\|}$ (since $\overline{C}^{w}$ is the smallest weakly closed set that contains C).

(4) By assumption $x \in \overline{\{x_n\colon n \geqslant 1\}}^{w} \subset \overline{conv\{x_n\colon n \geqslant 1\}}^{w} = \overline{conv\{x_n\colon n \geqslant 1\}}^{\|\cdot\|}$ (by (3)). Therefore, there is $(y_n) \in conv\{x_n\colon n \geqslant 1\}$ such that $y_n \to x$ in norm. But for any set A in a vector space, the convex hull of A satisfies $conv(A) = \{\sum_{i=1}^{n} \lambda_i a_i\colon n \geqslant 1, (a_i)_{i=1}^{n} \in A, (\lambda_i)_{i=1}^{n} \in [0, \infty), \sum_i^n \lambda_i = 1\}$ (one inclusion follows from a simple induction on the definition of convexity, for the other inclusion you need to show that the set on the right is convex).

□

Problem 3. *Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and define*

$$c_0(\mathbb{N}; \mathbb{F}) \stackrel{\text{def}}{=} \{\{x_n\}_{n=1}^{\infty} \subset \mathbb{F}\colon \lim_{n\to\infty} x_n = 0\}.$$

For $x = \{x_n\}_{n=1}^{\infty} \in c_0(\mathbb{N};\mathbb{F})$ *define* $\|x\|_\infty = \sup_{n\in\mathbb{N}} |x_n|$.

(1) *Show that* $(c_0(\mathbb{N};\mathbb{F}), \|\cdot\|_\infty)$ *is a normed vector space.*
(2) *Show that* $(c_0(\mathbb{N};\mathbb{F}), \|\cdot\|_\infty)$ *is a Banach space.*

Solution. It is clear that $\|\cdot\|$ is a norm on c_0.

Using a result of class (Theorem 7.1.4) we let $(x^{(n)})$ be a sequence in c_0, so that $\sum_{n=1}^{\infty} \|x^{(n)}\| < \infty$, and have to show that there is an $x \in c_0$ so that $\|\sum_{i=1}^{n} x^{(i)} - x\| \to 0$ if $n \to \infty$.

Write for $n \in \mathbb{N}$ $x^{(n)}$ as $(x_m^{(n)})_{m\in\mathbb{N}} \subset \mathbb{K}$ with $\lim_{m\to\infty} x_m^{(n)} = 0$. For fixed $m \in \mathbb{N}$ we notice that

$$\sum_{n\in\mathbb{N}} |x_m^{(n)}| \leqslant \sum_{n\in\mathbb{N}} \|x^{(n)}\| < \infty.$$

Since $\mathbb{K}$ is complete, we deduce that for each $m \in \mathbb{N}$ there is a number $y_m \in \mathbb{K}$ so that $y_m = \lim_{n\to\infty} \sum_{i=1}^{n} x_m^{(i)}$.

We have to show that $y = (y_m) \in c_0$ and that $\|y - \sum_{j=1}^{n} x^{(j)}\|_{c_0} \to 0$, for $n \to \infty$.

Let $\varepsilon > 0$ and we need to find $m \in \mathbb{N}$ so that $|y_k| < \varepsilon$ for all $k \geqslant m$ (this would prove that $y = (y_k) \in c_0$). First choose $N \in \mathbb{N}$ so that

$$\sum_{n=N+1}^{\infty} \|x^{(n)}\| < \varepsilon/2.$$

Then choose $m \in \mathbb{N}$, so that for all $k \geqslant m$ it follows that $|\sum_{n=1}^{N} x_k^{(n)}| \leqslant \varepsilon/2$.

Now note that for all $k \geqslant m$ we have:

$$\begin{aligned} |y_k| &\leqslant \Big|y_k - \sum_{n=1}^{N} x_k^{(n)}\Big| + \Big|\sum_{n=1}^{N} x_k^{(n)}\Big| \\ &= \Big|\sum_{n=N+1}^{\infty} x_k^{(n)}\Big| + \varepsilon/2 \\ &\leqslant \sum_{n=N+1}^{\infty} \|x^{(n)}\| + \varepsilon/2 < \varepsilon. \end{aligned}$$

Since we started out with an arbitrary $\varepsilon > 0$, and found an $m \in \mathbb{N}$ so that $|y_k| < \varepsilon$, for $k \geqslant m$, we showed that $x \in c_0$.

In order show that $\lim_{n\to\infty} \|y - \sum_{i=1}^{n} x^{(i)}\| = 0$ we let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ so that $\sum_{j=N}^{\infty} \|x^{(j)}\| < \varepsilon$. Then it follows for $n \geqslant N$

$$\Big\|y - \sum_{i=1}^{n} x^{(i)}\Big\| = \sup_{k\in\mathbb{N}} \Big|y_k - \sum_{i=1}^{n} x_k^{(i)}\Big| = \sup_{k\in\mathbb{N}} \Big|\sum_{j=n+1} x_k^{(j)}\Big| \leqslant \sum_{j=n+1} \|x^{(j)}\| < \varepsilon.$$

□

Problem 4. *Let* $(X, \mathcal{M}, \mu)$ *be a measure space and denote* $L_p(\mu) := L_p(X, \mathcal{M}, \mu)$ *for* $1 \leqslant p < \infty$.

(1) *If* $f \in L_2(\mu)$, $g \in L_3(\mu)$ *and* $h \in L_6(\mu)$, *show that* $fgh \in L_1(\mu)$ *and*

$$\|fgh\|_1 \leqslant \|f\|_2 \cdot \|g\|_3 \cdot \|h\|_6.$$

(2) *Formulate a generalization for the product of finitely many functions, and prove it.*

Solution. We note that $\frac{2}{3} + \frac{2}{6} = 1$. So applying the Hölder inequality to $p = \frac{3}{2}$ and $q = 3 = \frac{6}{2}$ and the functions $|g|^2$ and $|h|^2$ implies that

$$\begin{aligned}\left\| |g|^2 |h|^2 \right\| &\leqslant \left\| |g|^2 \right\|_{3/2} \cdot \left\| |h|^2 \right\|_3 \\ &= \left(\int |g|^3 \, d\mu \right)^{2/3} \cdot \left(\int |g|^3 \, d\mu \right)^{1/3} \\ &= \|g\|_3^2 \cdot \|h\|_6^2.\end{aligned}$$

This implies that $gh \in L_2(\mu)$ again applying the Hölder inequality to $p = q = \frac{1}{2}$ and the functions f and $|g||h|$ implies that

$$\|f \cdot g \cdot h\| \leqslant \|f\|_2 \leqslant \left\| hg \right\|_2 \leqslant \|f\|_2 \|g\|_3 \cdot \|h\|_2.$$

The generalization is the following:

Let $p_1, p_2, \ldots, p_n \in (1, \infty)$ and assume that

$$\frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_n} = 1$$

and if $f_i \in L_{p_i}(\mu)$ for $i = 1, 2, \ldots, n$, then the product $f = f_1 \cdot f_2 \cdot \ldots \cdot f_n$ is in $L_1(\mu)$ and

$$\|f_1 \cdot f_2 \cdot \ldots \cdot f_n\|_1 \leqslant \|f_1\|_{p_1} \cdot \|f_2\|_{p_2} \cdot \ldots \cdot \|f_n\|_{p_n}.$$

This can be easily proven by induction using the first part as induction step.

□

Problem 5. *Let X be an infinite dimensional normed vector space. Show that $\overline{S_X}^w = B_X$, i.e. the weak-closure of the unit sphere of X is the unit ball of X.*

Hint: Use Mazur's Theorem for one inclusion. For the other inclusion, show that every weak-neighborhood of a point $x_0 \in B_X \setminus S_X$ contains an affine line which eventually intersects the sphere.

Solution. Since $S_X \subset B_X$ then $\overline{S_X}^w \subset \overline{B_X}^w = B_X$ (since B_X is convex and norm closed and Mazur's theorem). For the other inclusion, let $x_0 \in B_X \setminus S_X$ and $V = \{x \in X : |x_i^*(x - x_0)| < \varepsilon, \forall 1 \leqslant i \leqslant n\}$ be a weak neighborhood of x_0 for some $x_1^*, \ldots, x_n^* \in X^*$ and $\varepsilon > 0$. The following claim was essentially proved in class when showing that any weakly open set is unbounded.

Claim 1. *There is $z_0 \in X \setminus \{0\}$ such that $x_0 + \mathbb{R} z_0 \subset V$.*

Assuming Claim 1, let $\varphi(t) = \|x_0 + t z_0\|$ for all $t \in \mathbb{R}$. It follows from the triangle inequality that φ is $\|z_0\|$-Lipschitz (and hence continuous). Moreover, $\varphi(0) = \|x_0\| < 1$ and $\varphi(t) \geqslant |t|\|z_0\| - \|x_0\| \to_{|t| \to \infty} \infty$. By the intermediate value theorem, there exists $t_0 \in \mathbb{R}$ such that $\varphi(t_0) = 1$, i.e. $\|x_0 + t_0 z_0\| = 1$. We have shown that $x_0 + t_0 z_0 \in S_X \cap V$, and hence $x_0 \in \overline{S_X}^w$, and hence $B_X = (B_X \setminus S_X) \cup S_X \subset \overline{S_X}^w$. □