REAL ANALYSIS MATH 608
HOMEWORK #8

Problem 1. *Let $(X, \mathcal{M}, \mu)$ be a measure space. Show that:*

(1) *For all $f \in L_1(\mu), g \in L_\infty(\mu)$ we have $fg \in L_1(\mu)$ and $\|fg\|_1 \leqslant \|f\|_1 \|g\|_\infty$, with equality if and only if $g = \|g\|_\infty$, μ-a.e. on $\{f \neq 0\}$.*
(2) *$\{f_n\}_n$ is a Cauchy sequence in $L_\infty(\mu)$ if and only if there is $A \in \mathcal{M}$ with $\mu(A^c) = 0$ such that $\lim_{n,k\to\infty} \sup_{x\in A} |f_n(x) - f_k(x)| = 0$.*
(3) *$L_\infty(\mu)$ is a Banach space.*

Problem 2. *Let $1 \leqslant p < q < r \leqslant \infty$, and consider the norm $\|f\|_{L_p \cap L_r} \stackrel{\text{def}}{=} \|f\|_p + \|f\|_r$ on $L_p \cap L_r$. Show that*

(1) *$(L_p \cap L_r, \|\cdot\|_{L_p\cap L_r})$ is a Banach space.*
(2) *Show that the formal inclusion $\iota\colon L_p \cap L_r \to L_q$ is continuous.*

Problem 3. *Let H be a Hilbert space, $S \subset H$, and recall that*

$$S^\perp = \{x \in H\colon\ x \perp y, \text{ for all } y \in S\}.$$

Show that $S^{\perp\perp}$ is the smallest closed linear subspace of H containing S.

Problem 4.

(1) *Show that if X is separable then the weak-$*$ topology on B_{X^*} is metrizable.*
(2) *Show that if X^* is separable then the weak topology on B_X is metrizable.*

Hint: For (1) consider $d(x^*, y^*) = \sum_{n=1}^{\infty} 2^{-n} |(x^* - y^*)(z_n)|$ where $\{z_n\}_n$ is dense in B_X. For (2) mimic the argument in (1).

Problem 5. *Let $1 \leqslant p < q < r \leqslant \infty$, and consider the norm on $L_p + L_r$ given by*

$$\|f\|_{L_p+L_r} \stackrel{\text{def}}{=} \inf\{\|g\|_p + \|h\|_r\colon\ f = g + h \in L_p + L_r\}.$$

Show that

(1) *$(L_p + L_r, \|\cdot\|_{L_p+L_r})$ is a Banach space.*
(2) *Show that the formal inclusion $\iota\colon L_q \to L_p + L_r$ is continuous.*