REAL ANALYSIS MATH 608
HOMEWORK #8

Problem 1. *Let $(X, \mathcal{M}, \mu)$ be a measure space. Show that:*

(1) For all $f \in L_1(\mu), g \in L_\infty(\mu)$ we have $fg \in L_1(\mu)$ and $\|fg\|_1 \leqslant \|f\|_1 \|g\|_\infty$, with equality if and only if $|g| = \|g\|_\infty$, μ-a.e. on $\{f \neq 0\}$.

(2) $\{f_n\}_n$ is a Cauchy sequence in $L_\infty(\mu)$ if and only if there is $A \in \mathcal{M}$ with $\mu(A^c) = 0$ such that

$$\lim_{n,k\to\infty} \sup_{x\in A} |f_n(x) - f_k(x)| = 0.$$

(3) $L_\infty(\mu)$ is a Banach space.

Solution. (1) $\int |fg| d\mu \leqslant \|g\|_\infty \int |f| d\mu = \|g\|_\infty \|f\|_1$. If $\mu(\{|g| < \|g\|_\infty\} \cap \{f \neq 0\}) > 0$ then $|f|(\|g\|_\infty - |g|)$ is a measurable function that is positive on a set of positive measure and thus $\int |f| \|g\|_\infty - |fg| d\mu > 0$, i.e. $\|fg\|_1 < \|f\|_1 \|g\|_\infty$. The sufficient condition is clear.

(2) For the necessary condition, let $\varepsilon > 0$, then $\{|f_n - f_k| > \varepsilon\} = (\{|f_n - f_k| > \varepsilon\} \cap A) \cup (\{|f_n - f_k| > \varepsilon\} \cap A^c)$ and hence $\mu(\{|f_n - f_k| > \varepsilon\}) \leqslant \mu(\{|f_n - f_k| > \varepsilon\} \cap A) + 0$. But there is $n_0 \geqslant 1$ such that for all $n, k \geqslant n_0$, $\sup_{x\in A} |f_n(x) - f_k(x)| < \varepsilon$, and thus $\mu(\{|f_n - f_k| > \varepsilon\}) = 0$, i.e. $\|f_n - f_k\|_\infty \leqslant \varepsilon$ whenever $n, k \geqslant n_0$. Assume now that $(f_n)_n$ is Cauchy for $\|\cdot\|_\infty$, and given $\varepsilon > 0$ let $n_0 \geqslant 1$ such that for all $n, k \geqslant n_0$ one has $\|f_n - f_k\|_\infty \leqslant \varepsilon$. Let $A_{n,k} \stackrel{\text{def}}{=} \{x \in X \colon |f_n(x) - f_k(x)| > \|f_n - f_k\|_\infty\}$ and $A = X \setminus \cup_{n,k} A_{n,k} = X \cap \cap_{n,k} (A^c_{n,k})$. For all $x \in A$, $n, k \geqslant 1$, $|f_n(x) - f_k(x)| \leqslant \|f_n - f_k\|_\infty$ and hence $(f_n - f_k)_{n,k}$ converges uniformly to 0 on A. Now, $\mu(\cup_{n,k} A_{n,k}) \leqslant \sum_{n,k} \mu(A_{n,k}) = 0$ since $\mu(A_{n,k}) = 0$ for all $n, k \geqslant 1$.

(3) If $(f_n)_n$ is Cauchy for the sup norm then by (2) it is Cauchy for the uniform norm outside a set of measure 0 and the completness of the sup norm follows from from completness of the uniform norm.

□

Problem 2. *Let $1 \leqslant p < q < r \leqslant \infty$, and consider the norm $\|f\|_{L_p \cap L_r} \stackrel{\text{def}}{=} \|f\|_p + \|f\|_r$ on $L_p \cap L_r$. Show that*

(1) $(L_p \cap L_r, \|\cdot\|_{L_p\cap L_r})$ is a Banach space.

(2) Show that the formal inclusion $\iota \colon L_p \cap L_r \to L_q$ is continuous.

Solution.

□

Problem 3. *Let H be a Hilbert space, $S \subset H$, and recall that*

$$S^\perp = \{x \in H \colon x \perp y, \text{ for all } y \in S\}.$$

Show that $S^{\perp\perp}$ is the smallest closed linear subspace of H containing S.

Solution. Let E be the smallest closed linear space that contains S. For any set S, $S^\perp$ is closed and linear. Indeed, if $x, y \in S^\perp$ then $< s, x + \lambda y >=< s, x > + \bar{\lambda} < s, y >= 0$ for all $s \in S$, and hence $x + \lambda y \in S^\perp$. If $(x_n)_n \subset S^\perp$ such that $x_n \to x$, then for all $s \in S$, $n \geqslant 1$ $< s, x_n >= 0$ and by continuity of the scalar product (CS inequality) we have that $< s, x >= 0$, i.e. $x \in S^\perp$. Observe now that $S \subset S^{\perp\perp}$. Indeed, if $s \in S$ and $y \in S^\perp$, then $< s, y >= 0$, and hence $S \subset (S^\perp)^\perp = S^{\perp\perp}$. Therefore by minimality of E one has $S \subset E \subset S^{\perp\perp}$. Assume now that $x \in S^{\perp\perp}$. If p_E denotes the orthogonal projection on E (which exists since E is a closed subspace of a Hilbert space), then $x = p_E(x) + x - p_E(x)$ and $p_E(x) \perp (x - p_E(x))$. We want to show that $x = p_E(x)$ and thus $x \in E$. But this is true as $\|x - p_E(x)\|^2 =< x - p_E(x), x - p_E(x) >=< x - p_E(x), x >= 0$ since $x \in S^{\perp\perp} \subset E^{\perp\perp}$ ($A \subset B \implies B^\perp \subset A^\perp$) and $x - p_E(x) \in E^\perp$. Thus we have $S \subset E = S^{\perp\perp}$

Remark 1. *Note that $E = \overline{span(S)} \stackrel{\text{def}}{=} \overline{\left\{\sum_{i=1}^n \lambda_i x_i : n \in \mathbb{N}, x_1, \ldots, x_n \in S, \lambda_1, \ldots, \lambda_n \in \mathbb{F}\right\}}$. Indeed $E \subset \overline{span(S)}$ (since the closure of a linear space is a linear space that is closed), but since E is a linear space containing S one has $span(S) \subset E$ (Since $span(S)$ is the smallest linear space containing S) and taking closure $\overline{span(S)} \subset E$ since E is closed. So in any normed vector space the closest linear space containing S is $\overline{span(S)}$.*

□

Problem 4.

(1) Show that if X is separable then the weak-$*$ topology on B_{X^*} is metrizable.
(2) Show that if X^* is separable then the weak topology on B_X is metrizable.

Hint: For (1) consider $d(x^*, y^*) = \sum_{n=1}^\infty 2^{-n} |(x^* - y^*)(z_n)|$ where $\{z_n\}_n$ is dense in B_X. For (2) mimic the argument in (1).

Solution.

(1) Since X is separable we can pick a sequence $\{z_n\}_n$ that is dense in B_X. For any $x^*, y^* \in B_{X^*}$ let $d(x^*, y^*) \stackrel{\text{def}}{=} \sum_{n=1}^\infty 2^{-n} |(x^* - y^*)(z_n)|$. It is clear that d is a symmetric map $B_{X^*} \times B_{X^*} \to [0, 2]$ that satisfies the triangle inequality. In order to show that two topologies are equivalent (i.e. have the same open sets) we need to verify that every neighborhood in a neighborhood basis for one topology contains a neighborhood for the other, and vice versa (remember that a set is open if it is a neighborhood of all its points). Given a ball $B_d(x^*, r)$ we need to find $x_1, \cdots, x_n \in X$ and $\varepsilon > 0$ such that $\{y^* \in B_{X^*} : |(x^* - y^*)(x_i)| < \varepsilon, 1 \leqslant i \leqslant n\} := V_{x^*, x_1, \ldots, x_n, \varepsilon} \subset B_d(x^*, r)$. If $N \geqslant 1$ is such that $\frac{1}{2^{N-1}} < \frac{r}{2}$, consider $y^* \in V_{x^*, z_1, \ldots, z_N, r/2}$, then

$$\begin{aligned}
\sum_{n=1}^\infty 2^{-n} |(x^* - y^*)(z_n)| &= \sum_{n=1}^N 2^{-n} |(x^* - y^*)(z_n)| + \sum_{n=N+1}^\infty 2^{-n} |(x^* - y^*)(z_n)| \\
&\leqslant \sum_{n=1}^N 2^{-n} \frac{r}{2} + \sum_{n=N}^\infty \frac{1}{2^n} \\
&< \frac{r}{2} + \frac{1}{2^{N-1}} < r,
\end{aligned}$$

and hence $V_{x^*, z_1, \ldots, z_N, r/N} \subset B_d(x^*, r)$.

Now consider $V_{x^*, x_1, \ldots, x_k, \varepsilon}$. Since $V_{x^*, \lambda x_1, \ldots, \lambda x_k, \lambda\varepsilon} = V_{x^*, x_1, \ldots, x_k, \varepsilon}$ for any $\lambda > 0$, one can assume that $\max_i \|x_i\| \leqslant 1$. For all $1 \leqslant i \leqslant k$ let z_{j_i} such that $\|z_{j_i} - x_i\| < \frac{\varepsilon}{4}$, and assume (after relabelling if needed) that $j_1 \leqslant j_2 \leqslant \ldots \leqslant j_k$. Then, if $y^* \in B_d(x^*, \frac{\varepsilon}{2^{j_k+1}})$ then $\sum_{n=1}^\infty 2^{-n} |(x^* - y^*)(z_n)| \leqslant \frac{\varepsilon}{2^{j_k+1}}$ and in particular for all $1 \leqslant n \leqslant j_k$, $|(x^* - y^*)(z_n)| < \frac{\varepsilon}{2} 2^{n - j_k} \leqslant \frac{\varepsilon}{2}$. Therefore, for all $1 \leqslant i \leqslant k$ one has $|(x^* - y^*)(x_i)| \leqslant |(x^* - y^*)(x_i - z_{j_i})| + |(x^* - y^*)(z_{j_i})| < 2\|x_i - z_{j_i}\| + \frac{\varepsilon}{2} < \varepsilon$, i.e. $B_d(x^*, \frac{\varepsilon}{2^{j_k+1}}) \subset V_{x^*, x_1, \ldots, x_k, \varepsilon}$.

(2) This is verbatim the same proof modulo swapping the role of X and X^*. Since X^* is separable we can pick a sequence $\{z_n^*\}_n$ that is dense in B_{X^*}. For any $x, y \in B_X$ let $d(x, y) \stackrel{\text{def}}{=} \sum_{n=1}^\infty 2^{-n} |z_n^*(x - y)|$. It is clear that d is a symmetric map $B_X \times B_X \to [0, 2]$ that satisfies the triangle inequality. In order to show that two topologies are equivalent (i.e. have the same open sets) we need to verify that every neighborhood in a neighborhood basis for one topology contains a neighborhood for the other, and vice versa (remember that a set is open if it is a neighborhood of all its points). Given a ball $B_d(x, r)$

we need to find $x_1^*, \cdots, x_n^* \in X^*$ and $\varepsilon > 0$ such that $\{y \in B_X : |x_i^*(x-y)| < \varepsilon, 1 \leqslant i \leqslant n\} := V_{x,x_1^*,\dots,x_n^*,\varepsilon} \subset B_d(x,r)$. If $N \geqslant 1$ is such that $\frac{1}{2^{N-1}} < \frac{r}{2}$, consider $y \in V_{x,z_1^*,\dots,z_N^*,r/2}$, then

$$\begin{aligned}\sum_{n=1}^{\infty} 2^{-n}|z_n^*(x-y)| &= \sum_{n=1}^{N} 2^{-n}|z_n^*(x-y)| + \sum_{n=N+1}^{\infty} 2^{-n}|z_n^*(x-y)| \\ &\leqslant \sum_{n=1}^{N} 2^{-n}\frac{r}{2} + \sum_{n=N}^{\infty}\frac{1}{2^n} \\ &< \frac{r}{2} + \frac{1}{2^{N-1}} < r,\end{aligned}$$

and hence $V_{x,z_1^*,\dots,z_N^*,r/N} \subset B_d(x,r)$.

Now consider $V_{x,x_1^*,\dots,x_k^*,\varepsilon}$. Since $V_{x,\lambda x_1^*,\dots,\lambda x_k^*,\lambda\varepsilon} = V_{x,x_1^*,\dots,x_k^*,\varepsilon}$ for any $\lambda > 0$, one can assume that $\max_i \|x_i^*\| \leqslant 1$. For all $1 \leqslant i \leqslant k$ let $z_{j_i}^*$ such that $\|z_{j_i}^* - x_i^*\| < \frac{\varepsilon}{4}$, and assume (after relabelling if needed) that $j_1 \leqslant j_2 \leqslant \ldots \leqslant j_k$. Then, if $y \in B_d(x, \frac{\varepsilon}{2^{j_k+1}})$ then $\sum_{n=1}^{\infty} 2^{-n}|z_n^*(x-y)| \leqslant \frac{\varepsilon}{2^{j_k+1}}$ and in particular for all $1 \leqslant n \leqslant j_k$, $|z_n^*(x-y)| < \frac{\varepsilon}{2}2^{n-j_k} \leqslant \frac{\varepsilon}{2}$. Therefore, for all $1 \leqslant i \leqslant k$ one has $|x_i^*(x-y)| \leqslant |(x_i^* - z_{j_i}^*)(x-y)| + |z_{j_i}^*(x-y)| < 2\|x_i^* - z_{j_i}^*\| + \frac{\varepsilon}{2} < \varepsilon$, i.e. $B_d(x, \frac{\varepsilon}{2^{j_k+1}}) \subset V_{x,x_1^*,\dots,x_k^*,\varepsilon}$.

□

Problem 5. *Let $1 \leqslant p < q < r \leqslant \infty$, and consider the norm on $L_p + L_r$ given by*

$$\|f\|_{L_p+L_r} \overset{\text{def}}{=} \inf\{\|g\|_p + \|h\|_r : f = g + h \in L_p + L_r\}.$$

Show that

(1) *$(L_p + L_r, \|\cdot\|_{L_p+L_r})$ is a Banach space.*
(2) *Show that the formal inclusion $\iota\colon L_q \to L_p + L_r$ is continuous.*

Solution.

□