REAL ANALYSIS MATH 608
HOMEWORK #9

Problem 1. *Show that*

1. *Every Hilbert space has an orthonormal basis.*
2. *Every separable Hilbert space has a countable orthonormal basis.*
3. *All the orthonormal bases of a Hilbert space have the same cardinality.*

Hint: For (2) Gramm-Schmidt procedure. For (3) Schroeder-Bernstein theorem.

Problem 2. *Let H be a Hilbert space.*

1. *Given $x, y \in H$, show that if for all $\alpha \in \mathbb{F}$, $\|x + \alpha y\| \geqslant \|x\|$ then $x \perp y$.*
2. *Let P be a bounded linear projection on H. Show that the following assertions are equivalent*
   - (a) *P is a orthogonal projection*
   - (b) *$< Px, y > = < x, Py >$, for all $x, y \in H$.*
   - (c) *$\|P\| = 1$*

Problem 3.

1. *Show that the dual of a Hilbert space is a Hilbert space.*
2. *Show that every Hilbert space is reflexive.*

Hint: Representation theorem.

Problem 4. *Let Y be a subspace of a Banach space X. Show that*

1. *If Y is topologically complemented, then Y is closed and admits a topological complement.*
2. *If Y is closed and admits a topological complement, then Y is topologically complemented.*

Hint: Closed Graph Theorem.

Problem 5. *Let $U : H_1 \to H_2$ be a surjective linear map. Show that U preserves the scalar products if and only if U is an isometric isomorphism.*

Hint: Polarization identity.