REAL ANALYSIS MATH 608
HOMEWORK #9

Problem 1. *Show that*

(1) Every Hilbert space has an orthonormal basis.
(2) Every separable Hilbert space has a countable orthonormal basis.
(3) All the orthonormal bases of a Hilbert space have the same cardinality.

Hint: (1) Zorn's lemma – (2) Gramm-Schmidt – (3) Bessel's inequality and Schroeder-Bernstein

Solution. (1) This is a typical Zorn's lemma argument. Let $\Theta \stackrel{\text{def}}{=} \{(u_i)_{i\in I} \text{ orthonormal collection in } H\}$, partially ordered by inclusion. Then $\Theta \neq \emptyset$ since $\{h/\|h\|\}$ is vacuously orthonormal whenever $h \neq 0$. Let $(C_i)_i$ be a chain in Θ, then $\cup_i C_i$ is an orthonormal collection which contains all the C_i's by definition. By Zorn's lemma there is a maximal orthonormal family $\{u_i\}_i$ which is one of the various characterizations of being an orthonormal basis.

(2) Let $\{h_n\}_n$ be a dense sequence in H. By discarding terms if needed one can assume without loss of generality that $\{h_n\}_n$ is a linearly independent sequence and that $span\{h_n \colon n \geqslant 1\}$ is dense. After applying the Gramm-Schmidt process to the sequence we obtain an orthonormal sequence whose linear span is dense. Such a sequence must be an orthonormal basis, for otherwise we can find $v \neq 0$ such that $< v, h_n >= 0$ for all $n \geqslant 1$, and hence $v \in [span\{h_n \colon n \geqslant 1\}]^{\perp}$. By continuity of the scalar product it follows that $v \in \overline{span\{h_n \colon n \geqslant 1\}}^{\perp} = H^{\perp} = \{0\}$; a contradiction.

(3) Let $\{u_i\}_{i\in I}$ and $\{v_j\}_{j\in J}$ be two orthonormal bases for a Hilbert space H. If I and J are finite sets then H is isometrically isomorphic to $\ell_2^{\text{card}(I)}$ and to $\ell_2^{\text{card}(J)}$. By a dimensionality argument, necessarily $\text{card}(I) = \text{card}(J)$. Assume now that I and J are infinite sets. If follows from Bessel's inequality that for all $i \in I$, the set $A_i \stackrel{\text{def}}{=} \{j \in J \colon < u_i, v_j >\neq 0\} \subset J$ is at most countable. Since $v_j \neq 0$ there must be a $i_j \in I$ such that $< u_{i_j}, v_j >\neq 0$ (since $\{u_i\}_i$ is an orthonormal basis). Thus $J = \cup_{i\in I} A_i$ and $\text{card}(J) \leqslant \text{card}(I) \times \text{card}(\mathbb{N}) = \text{card}(I)$ since $\text{card}(I) \geqslant \text{card}(\mathbb{N})$. Exchanging the role of the two bases we get that $\text{card}(I) \leqslant \text{card}(J)$. We just showed that $\text{card}(J) \leqslant \text{card}(I)$ and $\text{card}(I) \leqslant \text{card}(J)$, and it follows from Schroeder-Bernstein theorem that $\text{card}(I) = \text{card}(J)$.

□

Problem 2. *Let H be a Hilbert space.*

(1) Given $x, y \in H$, show that $x \perp y$ if and only if for all $\alpha \in \mathbb{F}$ $\|x + \alpha y\| \geqslant \|x\|$.
(2) Let P be a bounded linear projection on H. Show that the following assertions are equivalent
   (a) P is a orthogonal projection
   (b) $< Px, y >=< x, Py >$, for all $x, y \in H$.
   (c) $\|P\| = 1$

Solution. (1) Observe that if $< x, y >= 0$ then $\|x + \alpha y\|^2 = \|x\|^2 + |\alpha|^2\|y\|^2 \geqslant |\alpha|^2\|y\|^2$, and thus $\|x + \alpha y\| \geqslant |\alpha|\|y\| \geqslant \alpha\|y\|$. On the other hand, if for all α, $\|x + \alpha y\| \geqslant \|x\|$ then by taking square and developing them with scalar products we have that $-Re(\alpha < y, x >) \leqslant |\alpha|^2\|y\|^2$ and if $< y, x >\neq 0$ we can certainly find an α that will lead to a contradiction (make sure you understand why the argument works for complex scalars).

(2) Assume that P is an orthogonal projection, then

$$<Px,y>=<Px,Py>+<Px,y-Py>=<Px,Py>=<Px-x,Py>+<x,Py>=<x,Py>$$

where we used that $P(X)$ is orthogonal to $(I-P)(X)$ twice. If (b) holds then for all $x,y\in H$,

$$<Px,y-Py>=<Px,y>-<Px,Py>=<Px,y>-<P^2x,y>=<Px,y>-<Px,y>=0,$$

i.e. $P(X)$ is orthogonal to $(I-P)(X)$ and hence P is the orthogonal projection. Assume now that $||P||=1$. For any projection $||Px||=||P^2x||\leqslant||P^2||||x||\leqslant||P||^2||x||$ and thus $||P||\geqslant1$ (alternatively $||Px||=||x||$ for all $x\in P(X)$), and thus $||P||\geqslant1$ for any bounded projection. The point here is that $||P||\geqslant1$ characterizes orthogonal projections. If P is orthogonal then for all $x\in H$, $Px\perp(x-Px)$ and by (1) $||Px||\leqslant||Px+x-Px||=||x||$, i.e. $||P||\leqslant1$. Assuming now that $normP\leqslant1$, then for all α, we need to show that $Ker(P)=(I-P)(X)$ is orthogonal to $P(X)$. So let $y\in Ker(P)$, $z\in P(X)$ and $\lambda\in\mathbb{F}$, then $||z||=||Pz||=||P(z+\lambda y)||\leqslant||z+\lambda y||$, and by (1) it follows that $z\perp y$.

□

Problem 3.

(1) Show that the dual of a Hilbert space is a Hilbert space.
(2) Show that every Hilbert space is reflexive.

Hint: Representation theorem.

Solution. (1) The dual of H is the vector space of bounded linear functional on H equipped with the norm $||T||_{H^*}=\sup_{||x||\leqslant1}||Tx||$. There is no reason why $||\cdot||_{H^*}$ should come from a product scalar, but because of Riesz representation theorem it does! Indeed, let $J_1\colon H\to H^*$ such that $J_1(x)(y)=<y,x>_H$. Then by the representation theorem, J_1 is a bijection and and $J_1^{-1}\colon H^*\to H$ is an anti-linear isometry. On H^*, define

$$<u^*,v^*>_{J_1}=\overline{<J_1^{-1}(u^*),J_1^{-1}(u^*)>_H},$$

and because J_1^{-1} is anti-linear it defines a scalar product. We denote by $||\cdot||_{J_1}$ the norm induced by this scalar product. Then,

$$||u^*||_{J_1}^2=<u^*,u^*>_{J_1}=\overline{<J_1^{-1}(u^*),J_1^{-1}(u^*)>_H}=||J_1^{-1}(u^*)||_H^2=||u^*||_{H^*}^2$$

where we use in the last equality that J_1^{-1} is an isometry, and hence $||\cdot||_{H^*}$ is induced by a scalar product , i.e. H^* is a Hilbert space.

(2) We need to verify that the canonical isometric embedding $\delta\colon H\to H^{**}$ defined by $\delta(h)(y^*)=y^*(h)$ is surjective. Consider the operator $\Delta\colon H\to H^{**}$ given by $\Delta=J_2\circ J_1$ where $J_2\colon H^*\to H^{**}$ is the anti-linear surjective isometry given by Riesz representation theorem (which applies here since by (1) H^* is a Hilbert space), i.e. $J_2\colon(H^*,<\cdot,\cdot>_{J_1})\to H^{**}$ is such that $J_2(z^*)(y^*)=<y^*,z^*>_{J_1}$. We will see that $\delta=\Delta$ and hence δ will be surjective since Δ is surjective.

$$\Delta(h)(y^*)=(J_2\circ J_1)(h)(y^*)=(J_2(J_1(h)))(y^*)=<y^*,J_1(h)>_{J_1}=\overline{<J_1^{-1}(y^*),J_1^{-1}(J_1(h))>_H}=<h,J_1^{-1}(y^*)>_H=y^*(h).$$

□

Problem 4. *Let Y be a subspace of a Banach space X. Show that*

(1) If Y is topologically complemented, then Y is closed and admits a topological complement.
(2) If Y is closed and admits a topological complement, then Y is topologically complemented.

Hint: Closed Graph Theorem.

Solution. (1) Let $P\colon X \to Y$ such that P is a bounded linear projection. Then, $X = P(X) \oplus \ker(P)$ (as this is true for any linear projection). Since P is continuous, $I - P$ is also continuous, and thus $Y = P(X) = \ker(I - P)$ and $\ker(P)$ are closed, i.e. Y is closed subspace that admits a a topological complement.

(2) Assume that $X = Y \oplus Z$ with Y and Z closed. Then for all $x \in X$, x can be written uniquely as $x = y_x + z_x$ for some $y_x \in Y$ and $z_x \in Z$. The map $P\colon X \to Y$ defined by $P(x) = y_x$ is easily seen to be well defined, linear, and a projection (verify this). To show that it is continuous, it is sufficient to show that the graph of P is closed and then invoke the CGT. So assume that $x_n \to x$, and $Px_n \to y$. We need to show that $y = Px$. Since Y is closed and $(Px_n)_n$ is a sequence in Y, it follows that $y \in Y$. Since $x = y + (x - y)$ it remains to show that $x - y \in Z$, and thus by uniqueness of the decomposition we would have that $y = Px$. To show that $x - y \in Z$, observe that $x_n - Px_n \in \ker(P) = Z$ and since Z is closed it follows from the assumptions that $x - y = \lim_n x_n - Px_n \in Z$, and the proof is complete.

□

Problem 5. *Let $U\colon H_1 \to H_2$ be a (surjective) linear map. Show that U preserves the scalar products if and only if U is an isometry (isometric isomorphism).*

Hint: Polarization identity.

Solution. If U preserves the scalar product then clearly $\|Ux\|^2 =< Ux, Ux >=< x, x >= \|x\|^2$, and U is an isometry. Since being an isometry implies injectivity, U is thus an isometric isomorphism whenever it is surjective. Now if U is an isometry, then by the polarization identity (twice),

$$\begin{aligned}
< Ux, Uy > &= \frac{1}{4}(\|Ux + Uy\|^2 - \|Ux - Uy\|^2 + i\|Ux + iUy\|^2 - i\|Ux - iUy\|^2) \\
&= \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2) \\
&=< x, y >,
\end{aligned}$$

and U preserves the scalar product. □