Midterm Test

Instructions. This test is due on 10/29/13. You may get help on the test only from your instructor, and no one else. You may use other books, the web, etc. If you do so, quote the source.

1. (10 pts.) Use the Courant-Fischer mini-max theorem to show that \( \lambda_2 < 0 \) for the matrix

\[
A = \begin{pmatrix}
0 & 1 & 3 \\
1 & 0 & 2 \\
3 & 2 & 0
\end{pmatrix}.
\]

2. (15 pts.) Let \( A \) be an \( n \times n \) complex matrix that satisfies \( A^*A = AA^* \). Show that \( A \) is diagonalizable and that there is a unitary matrix \( U \) for which \( U^*AU = \Lambda = \text{diag} (\lambda_1, \ldots, \lambda_n) \).

3. Let \( f \) be continuous on \([0,1]\), with \( f(0) = f(1) = 0 \) and let \( s \in S^{1/n}(1,0) \) be the linear spline interpolant to \( s \), with knots at \( x_j = \frac{j}{n} \).

   (a) (15 pts.) Let \( \lambda \in \mathbb{R} \). Show that \( \left| \int_0^1 s(x)e^{i\lambda x}dx \right| \leq \frac{2n^2}{\lambda^2} \omega(f, 1/n) \).

   (b) (5 pts.) Use the previous part to show that \( \left| \int_0^1 f(x)e^{i\lambda x}dx \right| \leq \omega(f, 1/n) + \frac{2n^2}{\lambda^2} \omega(f, 1/n) \).

4. Let \( \{\phi_n(x)\}_{n=0}^{\infty} \) be a set of polynomials orthogonal with respect to a weight function \( w(x) \) on a domain \([a,b]\). Assume that the degree of \( \phi_n \) is \( n \), and that coefficient of \( x^n \) in \( \phi_n(x) \) is \( k_n > 0 \). In addition, suppose that the continuous functions are dense in \( L^2_w[a,b] = \{f : [a,b] \to \mathbb{C} : \int_a^b |f(x)|^2 w(x)dx < \infty\} \).

   (a) (5 pts.) Show that \( \phi_n \) is orthogonal to all polynomials of degree \( n - 1 \) or less.

   (b) (10 pts.) Show that \( \{\phi_n\}_{n=0}^{\infty} \) is complete in \( L^2_w[a,b] \).

   (c) (10 pts.) Show that the polynomials satisfy the recurrence relation \( \phi_{n+1}(x) = (A_n x + B_n)\phi_n(x) + C_n \phi_{n-1}(x) \). Find \( A_n \) in terms of the \( k_n \)’s.
5. Suppose that $f(\theta)$ is $2\pi$-periodic function in $C^m(\mathbb{R})$, and that $f^{(m+1)}$ is piecewise continuous and $2\pi$-periodic. Here $m > 0$ is a fixed integer. Let $c_k$ denote the $k^{th}$ (complex) Fourier coefficient for $f$, and let $c_k^{(j)}$ denote the $k^{th}$ (complex) Fourier coefficient for $f^{(j)}$.

(a) (5 pts.) Show that $c_k^{(j)} = (ik)^j c_k$, $j = 1, \ldots, m + 1$.

(b) (5 pts.) For $k \neq 0$, show that the Fourier coefficient $c_k$ satisfies the bound

$$|c_k| \leq \frac{1}{2\pi |k|^{m+1}} \|f^{(m+1)}\|_{L^1[0,2\pi]}.$$

(c) (10 pts.) Let $S_n(\theta) = \sum_{k=-n}^{n} c_k e^{ik\theta}$ be the $n^{th}$ partial sum of the Fourier series for $f$, $n \geq 1$. Show that both of these hold for $f$.

$$\|f - S_n\|_{L^2} \leq C \|f^{(m+1)}\|_{L^1} n^{-m + \frac{1}{2}} \text{ and } \|f - S_n\|_{C[0,2\pi]} \leq C' \|f^{(m+1)}\|_{L^1} n^{-m}.$$

(d) (10 pts.) Let $f(x)$ be the $2\pi$-periodic function that equals $x^2(2\pi-x)^2$ when $x \in [0,2\pi]$. Verify that $f$ satisfies the conditions above with $m = 1$. With the help of (a), calculate the Fourier coefficients for $f$. (Hint: look at $f''$.)