Final Exam

Instructions. This test is due on 12/11/13. You may get help on the test only from your instructor, and no one else. You may use my notes, other books, the web, etc. If you do so, quote the source. Use this test paper as a cover sheet when you turn in the exam.

Notation. \( H \) denotes a separable Hilbert space; \( B(H) \), the bounded linear operators on \( H \); \( C(H) \), the compact operators in \( B(H) \).

1. Let \( f \in C[0,2\pi] \). In assignment 5, if you scale the integrals involved, you derived the trapezoidal rule for numerically approximating \( \int_0^{2\pi} f(x) \, dx \).

   If \( a = 0, b = 2\pi \) and \( f \) is \( 2\pi \)-periodic, this has the form \( Q_{\text{trap}}(f) = \frac{2\pi}{n} \left( \sum_{k=0}^{n-1} f\left(\frac{2\pi k}{n}\right) \right) \). Let \( E_n = \left| \int_0^{2\pi} f(x) \, dx - Q_n(f) \right| \).

   (a) (5 pts.) Show that \( Q_{\text{trap}}(e^{ikx}) = \begin{cases} 0 & k \not\equiv 0 \mod n \\ 2\pi & k \equiv 0 \mod n \end{cases} \).

   (b) (10 pts.) Let \( f(x) \) be the \( 2\pi \)-periodic function that equals \( x^2(2\pi - x)^2 \) when \( x \in [0,2\pi] \). Show that \( \int_0^{2\pi} f(x) \, dx = 16\pi^5/15 \). Prove that \( E_n \leq Cn^{-4} \). (Hint: \( f = \frac{8\pi^4}{15} - \frac{24}{\pi} \sum_{k \neq 0} e^{ikx} k^{-4} \).)

   (c) (5 pts.) Use Matlab or some other program to plot \( \log(E_n) \) vs. \( \log(n) \), for \( n = 16, 64, 256, 1024 \). This should be a straight line. What is its slope? Does it agree with what you found in part (b)?

2. Let \( H \) be a complex Hilbert space, with inner product \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \| \), and let \( L \in B(H) \).

   (a) (5 pts.) Verify that \( \langle L(u + e^{i\alpha}v), u + e^{i\alpha}v \rangle - \langle L(u - e^{i\alpha}v), u - e^{i\alpha}v \rangle = 2e^{-i\alpha} \langle Lu, v \rangle + 2e^{i\alpha} \langle Lv, u \rangle \). (You will need this below.)

   (b) (10 pts.) Show that if \( L = L^* \), then \( \|L\| = \sup_{\|u\|=1} |\langle Lu, u \rangle| \).

   (c) (10 pts.) Show that if \( M = \sup_{\|u\|=1} |\langle Lu, u \rangle| \), then \( M \leq \|L\| \leq 2M \), whether or not \( L \) is self adjoint. Give an example that shows this result is false in a real Hilbert space.

3. (10 pts.) Let \( K \in C(H) \) be self adjoint. Show that the only possible limit point of the set of eigenvalues of \( K \) is 0; i.e., the non-zero eigenvalues of \( K \) are isolated.
4. (5 pts.) Let $K \in C(H)$ be self adjoint. Suppose the range of $K$ contains a dense subset of $H$, and that an o.n. basis has been chosen for the eigenspace of each nonzero eigenvalue of $K$. Show that the set of all of these eigenvectors form a complete, orthonormal set.

5. Let $H = L^2[0,1]$. Consider the boundary value problem,

$$Lu := \frac{d}{dx} \left( (1 + x) \frac{du}{dx} \right) = f(x), \ u(0) = 0, \ u'(1) = 0. \quad (1)$$

(a) (5 pts.) Find $G(x,y)$, the Green’s function for (1).

(b) (5 pts.) Let $Gf(x) = \int_0^1 G(x,y)f(y)dy$. Show that the range of $G$ contains a dense set in $H$. (Hint: show that every $v \in C^2[0,1]$ with support in $(0,1)$ is in the range of $G$. Explain why this is dense in $H$.)

(c) (5 pts.) Use it and the previous problem to show that the eigenfunctions for $\frac{d}{dx} \left( (1 + x) \frac{du}{dx} \right) + \lambda u = 0$, $u(0) = 0$, $u'(1) = 0$ form a complete orthogonal set.

6. Let $\| \cdot \|_{op}$ be the operator norm for $B(H)$.

(a) (5 pts.) Show that in $\| \cdot \|_{op}$, then $B(H)$ is a Banach space.

(b) (10 pts.) Consider the operator $L = I - \lambda M$, with $M \in B(H)$. Show that if $|\lambda| < \|M\|_{op}^{-1}$, then, in the operator norm, $\sum_{k=0}^{\infty} \lambda^k M^k = (I - \lambda M)^{-1}$. (This series is called a Neumann expansion for $(I - \lambda M)^{-1}$.)

7. (10 pts.) Show that $B, B^{-1}$ are in $B(H)$, and $K \in C(H)$, then the range of $L = B + \lambda K$ is closed.