Example 4.11, Chapter 4, Decomposition. (Narcowich, Math 414)

Let $f_2(x) = 2\phi(4x) + 2\phi(4x-1) + \phi(4x-2) - \phi(4x-3)$. (See figure 4.12 in the text.) Since $4 = 2^2$, $j = 2$ and $f_2 \in V_2$. We want to decompose f into its components in V_0, W_0 and W_1. The text already does this by using these formulas (eqns 4,10 and 4.11 in the text):

$$\begin{aligned} \phi(2^j x - 2k) &= (\phi(2^{j-1}x - k) + \psi(2^{j-1}x - k))/2, \\ \phi(2^j x - 2k - 1) &= (\phi(2^{j-1}x) - \psi(2^{j-1}x))/2. \end{aligned}$$

Here, we want to use the method involving coefficients.

Since f_2 is expressed in the basis $\{\phi(2^2x - k\}_{k=-\infty}^{\infty}$ the coefficients for the $j = 2$ level are

$$a_0^2 = 2, \quad a_1^2 = 2, \quad a_2^2 = 1, \quad a_3^2 = -1,$$

with all other a_k^2's being 0. By Theorem 4.12, we also have

$$a_\ell^{j-1} = \frac{a_{2\ell}^j + a_{2\ell+1}^j}{2} \quad \text{and} \quad b_\ell^{j-1} = \frac{a_{2\ell}^j - a_{2\ell+1}^j}{2}.$$

These formulas will allow us to obtain all lower level coefficients. First, let's find the decomposition of f into its V_1 and W_1 components. Because $a_k^2 = 0$ for $k < 0$ and $k > 3$, the only nonzero coefficients are a_0^1 and a_1^1. Using the formulas, we have $a_0^1 = \frac{2+2}{2} = 2$ and $a_1^1 = \frac{1-1}{2} = 0$. Also if $\ell < 0$ or $\ell > 1$, $b_\ell^1 = 0$. Again using the formulas above, we see that $b_0^1 = \frac{2-2}{2} = 0$ and $b_1^1 = \frac{1-(-1)}{2} = 1$. Thus the decomposition into V_1 and W_1 components is

$$f_2(x) = 2\underbrace{\phi(2x)}_{f_1} + \underbrace{\psi(2x-1)}_{w_1}.$$

Second, we need to decompose f_1 into $f_1 = f_0 + w_0$. Since all of the $a_\ell^1 = 0$, except for $\ell = 0$, the only $a_\ell^0 \neq 0$ is $a_0^0 = \frac{2+0}{2} = 1$. Similarly, all b_ℓ^0's are 0, except for b_0^0, which is $b_0^0 = \frac{2-0}{2} = 1$. It follows that

$$f_1(x) = \underbrace{\phi(x)}_{f_0} + \underbrace{\psi(x)}_{w_0}$$

Combining these results in the decomposition that we wanted:

$$f_2 = \underbrace{\phi(x)}_{f_0} + \underbrace{\psi(x)}_{w_0} + \underbrace{\psi(2x-1)}_{w_1}$$