MRA Throughout these notes, we will refer to the definition of an MRA: There exists a sequence of subspaces of $L^2(\mathbb{R})$, $\{V_j\}_{j=-\infty}^{\infty}$ and a function $\phi \in V_0$ such that these hold

1. Nested. $\cdots \subset V_{j-1} \subset V_j \subset V_{j+1} \subset \cdots$.
2. Density. $\overline{\cup_j V_j} = L^2(\mathbb{R})$.
3. Separation. $\cap_j V_j = \{0\}$.
4. Scaling. $f(x) \in V_j$ if and only if $f(2^{-j}x) \in V_0$.
5. Orthonormal basis property. $\{\phi(x-k\}_{k=-\infty}^{\infty}$ is a basis for V_0.

Notes for April 22, 2022. We started with doing two things. The first is this. If we let $\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k)$, then we showed that $\{\phi_{j,k}\}_{k=-\infty}^{\infty}$ is an orthonormal basis for V_j. To see that the functions are orthonormal, note that

$$\langle \phi_{j,k}, \phi_{j,m} \rangle = \int_{-\infty}^{\infty} 2^{j/2}\phi(2^j x - k) 2^{j/2}\phi(2^j x - m)dx.$$

Changing variables, $t = 2^j x$ gives us $\langle \phi_{j,k}, \phi_{j,m} \rangle = \int_{-\infty}^{\infty} \phi(t-k)\phi(t-m)dt$. Using the orthonormality property, $\langle \phi_{j,k}, \phi_{j,m} \rangle = \int_{-\infty}^{\infty} \phi(t-k)\phi(t-m)dt = \delta_{k,m}$. Consequently, the set is orthonormal. To be a basis, we must also show that it spans V_j. By the scaling property, $f(x) \in V_J$ implies that $f(2^{-j}x) \in V_0$. Since the $\phi(x-k)$'s form an orthogonal basis for V_0, we have that $f(2^{-j}x) = \sum_{k=-\infty}^{\infty} c_k \phi(x-k)$, where $c_k = \int_{-\infty}^{\infty} f(2^{-j}t)\phi(t-k)dt$. Replacing x above by $2^j x$ we see that

$$f(x) = \sum_{k=-\infty}^{\infty} c_k \phi(2^j x - k) = \sum_{k=-\infty}^{\infty} 2^{-j/2} c_k \underbrace{2^{j/2}\phi(2^j x - k)}_{\phi_{j,k}(x)}. \tag{1}$$

Thus $\{\phi_{j,k}\}_{k=-\infty}^{\infty}$ is an orthonormal basis for V_j. Moreover, in the integral defining c_k we may change variables from t to $\tau = 2^{-j}t$. Doing this gives us $c_k = 2^j \int_{-\infty}^{\infty} f(u)\phi(2^j u - k)du = 2^{j/2}\langle f, \phi_{j,k} \rangle$. By (1), we arrive at

$$f(x) = \sum_{k=-\infty}^{\infty} \underbrace{2^{-j/2}c_k}_{\langle f, \phi_{j,k} \rangle} \phi_{j,k}(x) = \sum_{k=-\infty}^{\infty} \langle f, \phi_{j,k} \rangle \phi_{j,k}(x) \tag{2}$$

We can also use the orthogonal set $\{\phi(2^j x - k\}_{k=-\infty}^{\infty}$ to get the series

$$f(x) = \sum_{k=-\infty}^{\infty} a_k^j \phi(2^j x - k), \ a_k^j = 2^j \int_{-\infty}^{\infty} f(t)\phi(2^j t - k)dt \qquad (3)$$

The second thing is to derive the scaling relation. Because $\phi_{1,k}$ is an o.n. basis for V_1, and because $\phi \in V_0 \subset V_1$, $\phi(x) \in V_1$, using the orthogonal form in (3) we arrive at the *scaling relation* or *two-scale relation*:

$$\phi(x) = \sum_{k=-\infty}^{\infty} p_k \phi(2x - k), \text{ where } p_k = 2\int_{-\infty}^{\infty} \phi(t) 2^{1/2} \phi(2t - k)dt. \qquad (4)$$

The p_k's are called the *scaling coefficients*.

Example 0.1. *Haar MRA.* The scaling function is $\phi(x) = \begin{cases} 1 & 0 \le x < 1 \\ 0 & \text{otherwise} \end{cases}$ and $V_j = \{f \in L^2(\mathbb{R}) : f(x) = \text{ constant on } 2^{-j}k \le x < 2^{-j}(k+1)\}$. We dealt with this in chapter 4. Since $\phi(x) = \phi(2x) + \phi(2x-1)$, we have $p_0 = 1$ and $p_1 = 1$ and all other p_k's are 0.

Example 0.2. *Shannon MRA.* The scaling function is $\phi(x) = \frac{\sin(\pi x}{\pi x} =:$ sinc(x) and V_j is the set of all band-limited functions for which $\mathrm{supp(\hat{f})} \subseteq [-2^{\mathrm{j}}\pi, 2^{\mathrm{j}}\pi]$. Nesting holds because f being in V_j means that $\mathrm{supp(\hat{f})} \subseteq [-2^{\mathrm{j}}\pi, 2^{\mathrm{j}}\pi] \subset [-2^{\mathrm{j+1}}\pi, 2^{\mathrm{j+1}}\pi]$. By definition, $f \in V_{j+1}$. We'll skip density and separation. To see that scaling holds, we begin by using the sampling theorem to write f in V_0, which is composed of all band-limited functions with $\Omega = \pi$, by $f(x) = \sum_{k=-\infty} f(k)\mathrm{sinc(x-k)}$. This ensures that $\{\mathrm{sinc(x-k)}\}_{\mathrm{k=-\infty}}^{\infty}$ spans V_0. In addition, carrying the same calculation as in Example 2.2 in the text, with the roles of t and λ reversed, we see that $\mathcal{F}[\mathrm{sinc(x)}](\lambda) = \frac{1}{\sqrt{2\pi}}\mathrm{b}(\lambda)$, where $b(\lambda)$ is the "box" function $b(\lambda) = \begin{cases} 1 & -\pi \le \lambda < \pi, \\ 0 & \text{otherwise} \end{cases}$. By property 6, pg. 102, $\mathcal{F}[\mathrm{sinc(x-k)}](\lambda)) = \frac{\mathrm{e}^{-\pi\lambda}}{\sqrt{2\pi}}\mathrm{b}(\lambda)$ and Plancheral's Theorem, we have

$$\begin{aligned} \int_{-\infty}^{\infty} \mathrm{sinc(x-k)sinc(x-m)dx} &= \int_{-\infty}^{\infty} \frac{e^{-\lambda k}}{\sqrt{2\pi}} b(\lambda) \overline{\frac{e^{-\lambda m}}{\sqrt{2\pi}} b(\lambda)} d\lambda \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{|b(\lambda)|^2}_{1} e^{i\lambda(m-k)} d\lambda \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\lambda(m-k)} d\lambda = \delta_{k,m}, \end{aligned}$$

and, consequently, $\{\mathrm{sinc(x-k)}\}_{-\infty}^{\infty}$ is an orthonormal set. We can use the sampling theorem to find the p_k's. Again, $\mathrm{sinc(x)} \in \mathrm{V}_0 \subset \mathrm{V}_1$. Doing so yields $\mathrm{sinc(x)} = \sum_{\mathrm{k}=-\infty}^{\infty} \mathrm{sinc(k\pi/(2\pi)sinc(2x-k)} = \sum_{\mathrm{k}=-\infty}^{\infty} \mathrm{sinc(k/2)sinc(2x-k)}$. From the series we see that $p_k = \mathrm{sinc(k/2)}$.

Notes for April 29, 2022. The wavelet spaces are defined in a way similar to the ones in the Haar MRA. Namely, W_j is the orthogonal complement to V_j in V_{j+1}, so $V_{j+1} = V_j \oplus W_j$. Let's look at $j = 0$. We are looking for $\psi \in W_0$ such that $\{\psi(x-k)\}_{k=-\infty}^{\infty}$ is an orthonormal basis for W_0. Like ϕ, $\psi(x) \in V_1$, so that it will have an expansion

$$\psi(x) = \sum_{-\infty}^{\infty} q_k \phi(2x-k),\ q_k = 2^{-1/2}\langle \psi, \phi_{1,k}\rangle = 2\int_{-\infty}^{\infty} \psi(t)\phi(2t-k)dt. \quad (5)$$

Since we are requiring V_j and W_j to be orthogonal, we have that, for all k, m,

$$\int_{-\infty}^{\infty} \phi(x-m)\psi(x-k)dx = 0.$$

As is the case, for all orthonormal bases,

$$\int_{-\infty}^{\infty} \phi(x-m)\psi(x-n)dx = \sum_{k=-\infty}^{\infty} \langle \phi(t-m), \phi_{1,k}(t)\rangle \langle \psi(t-n), \phi_{1,k}(t)\rangle \quad (6)$$

We can compute the inner products above. To begin,

$$\begin{aligned}\langle \phi(t-m), \phi_{1,k}(t)\rangle &= 2^{1/2}\int_{-\infty}^{\infty} \phi(t-m)\phi(2t-k)dt \\ &= 2^{1/2}\int_{-\infty}^{\infty} \phi(t)\phi(2t-(k-2m))dt \\ &= 2^{-1/2}p_{k-2m}\end{aligned}$$

Similarly, $\langle \psi(t-n), \phi_{1,k}(t)\rangle = 2^{-1/2}q_{k-2n}$. Then, from (6), we have, for all m, n,

$$\frac{1}{2}\sum_{k=-\infty}^{\infty} p_{k-2m}q_{k-2n} = 0.$$

Using the trick of reversing[1] the p_k's and alternating the signs of them (see pg. 199 in the text), we get $q_k = (-1)^k p_{1-k}$.

Next, we will discuss the decomposition and reconstruction formulas. If $f \in V_{j+1}$, then, because $\{\phi_{j+1,k}\}_{k=-\infty}^{\infty}$ is an orthonormal basis for V_{j+1}, it has the expansion

$$f(x) = \sum_{k=-\infty}^{\infty} \langle f, \phi_{j+1} \rangle \phi_{j+1}(x) = \sum_{k=-\infty}^{\infty} a_k^{j+1} \phi(2^{j+1}x - k), \tag{7}$$

where $a_k^{j+1} = 2^{j+1} \int_{-\infty}^{\infty} f(t)\phi(2^{j+1}t - k)dt$. Because we also have that

$$\{\phi_{j,k}\}_{k=-\infty}^{\infty} \cup \{\psi_{j,k}\}_{k=-\infty}^{\infty}$$

is a basis for V_{j+1}, we can write f as

$$f(x) = \sum_{k=-\infty}^{\infty} a_k^j \phi(2^j x - k) + \sum_{k=-\infty}^{\infty} b_k^j \psi(2^j x - k)$$

We may use these to derive the formulas relating the a_j^{j+1}'s and the a_k^j's and b_k^j's. This is done in the text in section 5.1.4. The formulas are

$$\textsf{Decomposition:} \begin{cases} a_m^j &= 2^{-1} \sum_{k=-\infty}^{\infty} p_{k-2m} a_k^{j+1} \\ b_m^j &= 2^{-1} \sum_{k=-\infty}^{\infty} (-1)^k p_{1-k+2m} a_k^{j+1} \end{cases} \tag{8}$$

$$\textsf{Reconstruction:} \begin{cases} a_k^{j+1} &= \sum_{m=-\infty}^{\infty} p_{k-2m} a_m^j \\ &+ \sum_{m=-\infty}^{\infty} (-1)^k \, p_{1-k+2m} \, b_m^j \end{cases} \tag{9}$$

These formulas may be restated in terms of filters. Let $h_k = \frac{1}{2} p_{k+1}$, $\ell_k = \frac{1}{2} p_{-k}$, $\tilde{h}_k = (-1)^k p_{1-k}$ and $\tilde{\ell}_k = p_k$. Then, $a_m^J = (\ell * a^{j+1})_{2m}$, $b_m^j = (h * a^{j+1})_{2m}$, $a_k^{j+1} = \sum_{m=-\infty}^{\infty} \tilde{\ell}_{k-2m} a_m^j + \sum_{m=-\infty}^{\infty} \tilde{h}_{k-2m}, b_m^j$. Using down-sampling and up-sampling operators D and U, $a^j = D(\ell * a^{j+1}$, $b^j = D(h * a^{j+1})$, and $a^{j+1} = \tilde{\ell} * (Ua^j) + \tilde{h} * (Ub^j)$.

It is clear that the scaling coefficents, the p_k's, play a funcdamental role in the wavelet analysis described above. What if we don't know either the scaling function ϕ or the scaling spaces V_j but we do the p_k's. Is it possible

[1] E.g., if $x = [1 \quad 2 \quad -5 \quad -2 \quad 6 \quad -7]$ and $y = [-7 \quad -6 \quad -2 \quad 5 \quad 2 \quad -1]$, then $x \cdot y = 1 \cdot (-7) + (-1) \cdot (-7) + 2 \cdot (-6) + 6 \cdot 2 + (-5) \cdot (-2) + (-2) \cdot 5 = 0$.

to get ϕ and the V_j's from knowing the p_k's? Ingrid Daubechies adressed just this question. We will address this next class.

Notes for May 2, 2022

To keep matters simple, we address the case in which we have just four scaling coefficients, p_0, p_1, p_2 and p_3. We will suppose that we these coefficients, but not the scaling function ϕ. Even so, we can still write out the scaling relation, $\phi(x) = \sum_{k=0}^{3} p_k \phi(2x-k)$. If we take the Fourier transform of both sides of this equation, we get $\mathcal{F}[\phi(x)](\xi) = \sum_{k=0}^{3} p_k \mathcal{F}[\phi(2x-k)](\xi)$. Using eq. 2.7 in the text, $\mathcal{F}[\phi(2x-k)](\xi) = \frac{1}{2}e^{-ik/2}\mathcal{F}[\phi(x)](\xi/2)$, or, equivalently, $\mathcal{F}[\phi(2x-k)](\xi) = \frac{1}{2}e^{-ik/2}\widehat{\phi}(\xi/2)$. Hence, $\widehat{\phi}(\xi) = \sum_{k=0}^{3} \frac{1}{2}p_k e^{-ik\xi/2}\widehat{\phi}(\xi/2) = \big(\sum_{k=0}^{3} \frac{1}{2}p_k e^{-ik\xi/2}\big)\widehat{\phi}(\xi/2)$. Define the polynomial

$$P(z) = \sum_{k}^{3} p_k z^k = \frac{1}{2}(p_0 + p_1 z + p_2 z^2 + p_3 z^3).$$

The formula that we derived above then becomes

$$\widehat{\phi}(\xi) = P(e^{-i\xi/2})\widehat{\phi}(\xi/2).$$

Notice that $\phi(\xi/2) = P(e^{-i\xi/4})\widehat{\phi}(\xi/4)$, $\phi(\xi/4) = P(e^{-i\xi/8})\widehat{\phi}(\xi/8)$, and so on. If we use this in conjenction with the previous formula, we see that

$$\widehat{\phi}(\xi) = P(e^{-i\xi/2})P(e^{-i\xi/2^2})P(e^{-i\xi/2^3})\widehat{\phi}(\xi/2^3).$$

In paricular, we may write in product form:

$$\widehat{\phi}(\xi) = \big(\prod_{r=1}^{3} P(e^{-i\xi/2^r})\big)\widehat{\phi}(\xi/2^3)$$

Nothing stops us from adding more terms. If we use n terms, we get

$$\widehat{\phi}(\xi) = \big(\prod_{r=1}^{n} P(e^{-i\xi/2^r})\big)\widehat{\phi}(\xi/2^n) \qquad (10)$$

Taking the limit as $n \to \infty$ we get two things. First, $\lim_{n\to\infty}\widehat{\phi}(\xi/2^n) = \widehat{\phi}(\xi \lim_{n\to\infty} 2^{-n}) = \widehat{\phi}(0)$. It turns out that one can show $\widehat{\phi}(0) = \frac{1}{\sqrt{2\pi}}$, using only some normalizaton properties valid for any MRA. Taking the limit as $n \to \infty$ in (10) then results in

$$\widehat{\phi}(\xi) = \frac{1}{\sqrt{2\pi}} \prod_{r=1}^{\infty} P(e^{-i\xi/2^r}) \tag{11}$$

This is very important. The polynomial P doesn't depend on ϕ. The formula above gives the Fourier transform of ϕ , and hence ϕ itself, as well as the V_j's knowing only the p_k's plus the fact that they come from an MRA. Fly in the ointment: How do we know that the p_k's come from an MRA?