Examples using inner products – notes for 1/17/2020

1. Let $\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}dx$. For $f(x) = 3x - 1$ and $g(x) = \sqrt{x}$. Find the following quantities.

   (a) $\|f\|$ and $\|g\|$.

   (b) The angle θ between f and g.

   (c) The distance between f and g, $\|f - g\|$.

   Solution.

   (a) $\|f\|^2 = \int_0^1 f(x)^2dx = \int_0^1 (3x-1)^2dx = \int_0^1 (9x^2 - 6x + 1)dx = 3 - 3 + 1 = 1$. Hence, $\|f\| = 1$. For g, $\|g\|^2 = \int_0^1 (\sqrt{x})^2dx = \int_0^1 xdx = 1/2$, so $\|g\| = \frac{1}{\sqrt{2}}$.

   (b) We need $\langle f, g\rangle = \int_0^1 (3x-1)x^{1/2}dx = 3/(5/2) - 1/(3/2) = 6/5 - 2/3 = 8/15$. Thus, $\cos(\theta) = \frac{8/15}{1\cdot 2^{-1/2}} = 8\sqrt{2}/15$ and $\theta \approx 0.7163$ radians.

   (c) $\|f-g\|^2 = \int_0^1 (3x - 1 - x^{1/2})^2dx = \int_0^1 (9x^2 + x + 1 - 6x^{3/2} - 6x - 2x^{1/2})dx$. Doing the integral results in $\|f - g\| = \sqrt{13/30}$.

2. Consider the 2π periodic signal $f(t) = 1 + 2\sin(t) - \cos(4t)$. Find the energy in the signal over the period $-\pi \le t \le \pi$.

   Solution. The energy is $E = \int_{-\pi}^{\pi}(1 + 2\sin(t) - \cos(4t))^2dt$. Expanding this out we have

$$E = \int_{-\pi}^{\pi} (1 + 4\sin^2(t) + \cos^2(4t) + 4\sin(t) - 2\cos(4t) - 4\sin(t)\cos(4t))dt.$$

   Doing the integrals gives us $E = 2\pi + 4\pi + \pi + 0 + 0 + 0 = 7\pi$.

3. Let $\psi(t) := \begin{cases} 1 & 0 \le t < 1/2, \\ -1 & 1/2 \le t \le 1 \end{cases}$. The function is shown in Figure 1 below. Consider the space $L^2[0, 1]$. Let $f(t) = t^2$. Find $\langle \psi, f\rangle$

   Solution. $\langle \psi, f\rangle = \int_0^1 \psi(t)t^2dt$. Because ψ is discontinuous at $t = 1/2$, we have to break the integral into two pieces:

$$\int_0^1 \psi(t)t^2dt = \int_0^{1/2} \psi(t)t^2dt + \int_{1/2}^1 \psi(t)t^2dt = \int_0^{1/2} t^2dt - \int_{1/2}^1 t^2dt.$$

   Evaluating the integrals then gives $\langle \psi, f\rangle = -1/4$.

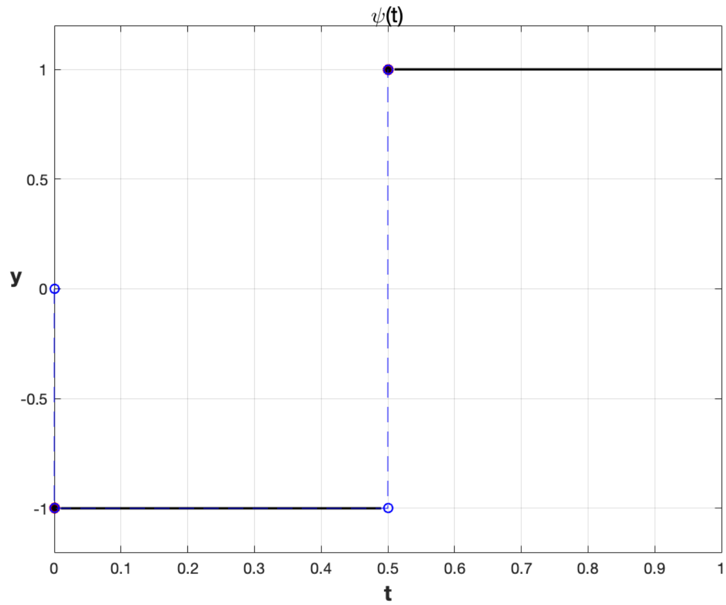

Figure 1: The Haar Wavelet, $\psi(t)$

For ordinary vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$, the projection of $\mathbf{v}$ onto a unit vector $\mathbf{u}$ is given by $\mathbf{p} = |\mathbf{v}|\cos(\theta)\mathbf{u}$, where θ is the usual angle between the two vectors. It represents the component of the vector $\mathbf{v}$ parallel to $\mathbf{u}$. In addition, the vector $\mathbf{q} = \mathbf{v} - \mathbf{p}$ is the component of $\mathbf{v}$ orthogonal to $\mathbf{u}$

The interpretation is the same in any inner product space[1], including L^2. If g and f are in $L^2[a,b]$, with $\|f\| = 1$, $P = \|g\|\cos(\theta)f = \langle g, f\rangle f$ is the projection of g onto f. In addition, $Q = g - P$ is the component of g orthogonal (perpendicular) to f.

Example. Find the projection of $g(x) = x$ onto $\frac{sin(2x)}{\sqrt{\pi}}$ in $L^2[-\pi, \pi]$. Solution.

$$P(x) = \left(\int_{-\pi}^{\pi} t\frac{\sin(2t)}{\sqrt{\pi}}dt\right)\frac{\sin(2x)}{\sqrt{\pi}} = -\sin(2x).$$

[1]This works for a complex vector space, provided the order $\langle g, f\rangle$ is kept.