C4

Part 2, #4

(a) $h(t) = 0 \quad \forall\, t<0 \Rightarrow L$ is Causal.

(b) $L[f] = \int_{-\infty}^{\infty} h(t-\tau)\, f(\tau)\, d\tau$. $\quad h(t-\tau) = \begin{cases} \frac{1}{d} & \text{if } 0 \le t-\tau \le d \\ 0 & \text{~~otherwise~~ otherwise.} \end{cases}$

$\Rightarrow L[f] = \frac{1}{d}\int_{t-d}^{t} f(\tau)\, d\tau,$

Three cases: (i) $t<0$, $f(\tau) = 0 \;\forall\, \tau \le 0 \Rightarrow L[f] = 0$.

(2) $t-d \le 0 \le t$: $f(\tau) = 0$ in $t-d \le \tau \le 0$ and $e^{-\tau}$ in $0 \le \tau \le t$.

$$\Rightarrow L[f] = \frac{1}{d}\int_0^t e^{-\tau}\, d\tau = \frac{1}{d}\left(1-e^{-t}\right).$$

(3) $0 \le t-d \Rightarrow t \ge d$. For all $\tau \ge d$, $f(\tau) = e^{-\tau}$.

$$\Rightarrow L[f] = \frac{1}{d}\int_{t-d}^{t} \cancel{e^{\tau}}\, e^{-\tau}\, d\tau = \frac{1}{d}\left(e^{-(t-d)} - e^{-t}\right).$$

(c) Convolution theorem

$$\widehat{L[f]} = \widehat{f * \cancel{h}} = \sqrt{2\pi}\,\hat{h}\,\hat{f}$$

$$\Rightarrow \widehat{L[f]} = \frac{\sqrt{2\pi}}{\sqrt{2\pi}}\left(\frac{1}{1+i\lambda}\right) \times \frac{1-e^{-i\lambda d}}{\sqrt{2\pi}\, i\lambda d}$$

$$\Rightarrow \widehat{L[f]} = \frac{1}{\sqrt{2\pi}\, i\lambda d}\left(\frac{1-e^{-i\lambda d}}{1+i\lambda}\right)$$